Advanced Linear Algebra II - HW1

甘

习题 5.0.6. 考虑列向量空间 K^4 中的向量

$$\alpha_1 = (1,\ 1,\ 0,\ 0)^T,\ \alpha_2 = (0,\ 1,\ 1,\ 0)^T \quad 和 \quad \beta_1 = (1,\ 2,\ 3,\ 4)^T,\ \beta_2 = (0,\ 1,\ 2,\ 2)^T.$$

令 $U = \operatorname{span}(\alpha_1,\ \alpha_2)$, $W = \operatorname{span}(\beta_1,\ \beta_2)$. 分别求 $U \cap W$ 和 $U + W$ 的一组基.

Pf. Let $\gamma \in U \cap W$. Then there exist $a_1, a_2, b_1, b_2 \in K$ s.t.

$$a_1\alpha_1 + a_2\alpha_2 = b_1\beta_1 + b_2\beta_2$$

Namely, $(a_1, a_2, -b_1, -b_2)^T$ is a solution of the system of linear eqns:

$$\begin{pmatrix} 1 & 0 & 1 & 0 \\ 1 & 1 & 2 & 1 \\ 0 & 1 & 3 & 2 \\ 0 & 0 & 4 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = 0$$

Denote the coef matrix by A. By Gauss reduction, it is equivalent to

$$\begin{pmatrix} 1 & 0 & 1 & 0 \\ & 1 & 1 & 1 \\ & & 2 & 1 \\ & & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = 0$$

Then $(a_1, a_2, -b_1, -b_2)^T \in \mathbb{C}\,(1, -1, -1, 2)^T$

Thus $U \cap W$ has a basis $\alpha_1 - \alpha_2 = (1, 0, -1, 0)^T$.

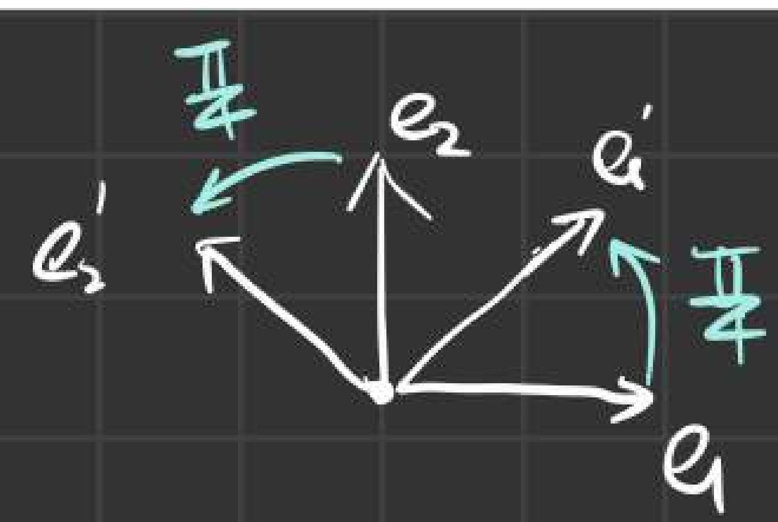
By column transformations, A is column equivalent to

$$\begin{pmatrix} 0 & 1 & 1 & 0 \\ & 1 & 0 & 1 \\ & 0 & -1 & 2 \\ & & 0 & 2 \end{pmatrix}$$

Thus $U + W$ has a basis $\alpha_2, \beta_1, \beta_2$.

习题 5.0.12. 写出实向量空间 $\mathbb{R}^2$ 上的满足 $\mathscr{A}^4 = -I$ 的一个线性变换 $\mathscr{A}$. (这里 I 表示恒等变换.)

Pf.



Let $\mathscr{A}$ be a rotation by $\frac{\pi}{4}$.

$$\mathscr{A}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{pmatrix}$$

习题 5.0.13. 对于一个复矩阵 $A = (a_{ij}) \in \mathbf{M}_{m\times n}(\mathbb{C})$, 以 $\overline{A}^T = (\overline{a}_{ji}) \in \mathbf{M}_{n\times m}(\mathbb{C})$ 表示它的**共轭转置**. 即, $\overline{A}^T$ 的第 (i, j) 位元素是 a_{ji} 的复共轭, 其中 a_{ji} 是 A 的第 (j, i) 位元素 (参见定义 8.1.3).

1. 按照通常的矩阵加法和数乘, 集合 $\mathbf{M}_{m\times n}(\mathbb{C})$ 作为 $\mathbb{C}$ 上向量空间的维数 $\dim_{\mathbb{C}} \mathbf{M}_{m\times n}(\mathbb{C})$ 是多少？若将 $\mathbf{M}_{m\times n}(\mathbb{C})$ 视为 $\mathbb{R}$ 上的向量空间, 其维数 $\dim_{\mathbb{R}} \mathbf{M}_{m\times n}(\mathbb{C})$ 是多少？

   如果一个方阵 $A \in \mathbf{M}_n(\mathbb{C})$ 满足 $A = \overline{A}^T$, 则称 A 是一个 Hermite **矩阵** (参见定义 8.1.3). 令 H 表示 $\mathbf{M}_n(\mathbb{C})$ 中所有 Hermite 矩阵构成的子集.

2. 若将 $\mathbf{M}_n(\mathbb{C})$ 视为复向量空间, H 是否是其子空间？若是, $\dim_{\mathbb{C}} H$ 是多少？若否, 原因是什么？

3. 若将 $\mathbf{M}_n(\mathbb{C})$ 视为实向量空间, H 是否是其子空间？若是, $\dim_{\mathbb{R}} H$ 是多少？若否, 原因是什么？

1. $\dim_{\mathbb{C}} M_{m\times n}(\mathbb{C}) = mn$. $\dim_{\mathbb{R}} M_{m\times n}(\mathbb{C}) = 2mn$

2. No. Let $A \in H$ be a nonzero matrix.

   $\overline{(iA)}^T = -i\,\overline{A}^T = -iA \neq iA$, where $i = \sqrt{-1}$.

3. Yes. Let $A, B \in H$. $a \in \mathbb{R}$

   ① $0 \in H$. ← Guarantee nonempty.

   ② $\overline{A+B}^T = (\overline{A} + \overline{B})^T = \overline{A}^T + \overline{B}^T = A + B$

   ③ $\overline{aA}^T = a\,\overline{A}^T = aA$

$\dim_{\mathbb{R}} H = n^2$. H has a $\mathbb{R}$-basis $E_{pq} + E_{qp}$, $i(E_{pq} - E_{qp})$, E_{rr}

for $1 \leq p < q \leq n$, $1 \leq r \leq n$.

习题 5.0.15. 设 $V = K[X]_{\leq 4}$, $U = \{f \in V \mid f(0) = f(1) = f(-1)\}$.

1. 找出 U 的一组基.
2. 将前一小题找到的 U 的基扩充为 V 的一组基.

Pf. 1. $1, \ x(x-1)(x+1), \ x^2(x-1)(x+1)$ is a basis of U

Obviously, these three are linearly independent and are in U.

U is a solution space of $\begin{cases} f(0) - f(1) = 0 \\ f(1) - f(-1) = 0 \end{cases}$, thus 3-dim.

2. Add x, x^2

习题 5.0.18. 考虑线性变换

$$\mathscr{A} : K^3 \longrightarrow K^3 ; \quad \begin{pmatrix} x \\ y \\ z \end{pmatrix} \longmapsto \begin{pmatrix} x - z \\ -x + 2y \\ y + z \end{pmatrix}.$$

1. 求 $\mathscr{A}$ 在有序标准基 $\mathcal{B} = (e_1, e_2, e_3)$ 下的矩阵.
2. 求 $\mathscr{A}$ 在有序基 (η_1, η_2, η_3) 下的矩阵, 其中

$$\eta_1 = (2, 0, -1)^T, \ \eta_2 = (1, 1, 1)^T, \ \eta_3 = (1, 0, 1)^T.$$

1. $\mathscr{A}e_1 = \begin{pmatrix} 1 \\ -1 \\ \end{pmatrix}, \quad \mathscr{A}e_2 = \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix}, \quad \mathscr{A}e_3 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$

$\Rightarrow \mathscr{A}(e_1, e_2, e_3) = (e_1, e_2, e_3) \begin{pmatrix} 1 & 0 & -1 \\ -1 & 2 & 0 \\ 0 & 1 & 1 \end{pmatrix}$

2. $\mathscr{A}\eta_1 = \mathscr{A}\begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix} = \begin{pmatrix} 3 \\ -2 \\ -1 \end{pmatrix} = (\eta_1, \eta_2, \eta_3)\begin{pmatrix} \frac{4}{3} \\ -2 \\ \frac{7}{3} \end{pmatrix}$

$\mathscr{A}\eta_2 = \mathscr{A}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} = (\eta_1, \eta_2, \eta_3)\begin{pmatrix} -\frac{2}{3} \\ 1 \\ \frac{1}{3} \end{pmatrix}$

$\mathscr{A}\eta_3 = \mathscr{A}\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} = (\eta_1, \eta_2, \eta_3)\begin{pmatrix} -\frac{1}{3} \\ -1 \\ \frac{5}{3} \end{pmatrix}$

$\Rightarrow M_\eta(\mathscr{A}) = \begin{pmatrix} \frac{4}{3} & -\frac{2}{3} & -\frac{1}{3} \\ -2 & 1 & -1 \\ \frac{7}{3} & \frac{1}{3} & \frac{5}{3} \end{pmatrix}$

$$\left(\begin{pmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ -1 & 1 & 1 \end{pmatrix} \underline{X} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \Rightarrow \underline{X} = \begin{pmatrix} \frac{1}{3}a_1 - \frac{1}{3}a_3 \\ a_2 \\ \frac{1}{3}a_1 - a_2 + \frac{2}{3}a_3 \end{pmatrix} \right)$$

习题 5.0.24. 设 $\mathscr{A}$ 为向量空间 V 上的线性变换.

1. 假设存在非零向量 v, $w \in V$ 满足 $\mathscr{A}v = 3w$, $\mathscr{A}w = 3v$. 证明: 3 或 -3 是 $\mathscr{A}$ 的特征值.
2. 证明: 如果 V 中的非零向量都是 $\mathscr{A}$ 的特征向量, 那么 $\mathscr{A}$ 一定是恒等变换 I 的常数倍.
3. 假设 V 是有限维的, $n = \dim V \geq 1$. 证明: 如果 V 的每个 $n-1$ 维子空间都是 $\mathscr{A}$ 的不变子空间, 那么 $\mathscr{A}$ 一定是恒等变换 I 的常数倍.

1. $\mathscr{A}(v+w) = 3(v+w)$, $\mathscr{A}(v-w) = -3(v-w)$

   $v+w$ & $v-w$ can not be both 0.

2. If not, then $\exists\ v, w \in V$ and $v, w \neq 0$ s.t.

   $\mathscr{A}v = \lambda_1 v$, $\mathscr{A}w = \lambda_2 w$ with $\lambda_1 \neq \lambda_2$.

   Note that v, w are linearly independent (in diff. eigenspaces)

   Thus, $\mathscr{A}(v-w) = \lambda_1 v - \lambda_2 w$ is NOT an eigenvector.

   Contradiction.

3. Suppose $\exists\ v_1 \in V$ s.t. $\mathscr{A}v_1 = v_2$ and v_1 are not linearly dependent.

   Extend v_1, v_2 to be a basis $v_1, v_2, \cdots, v_n$ of V.

   Then $\operatorname{span}\{v_1, v_3, v_4, \cdots, v_n\} \subseteq V$ is NOT $\mathscr{A}$-invariant.

   Contradiction! Thus, $\forall v \in V$, $\exists \lambda \in K$ s.t. $\mathscr{A}v = \lambda v$ for some λ.

   Then by 2, we are done.

习题 5.0.26. 设 $\mathscr{A}$ 是向量空间 V 上的线性变换. 假设 $\dim \operatorname{Im}(\mathscr{A}) = m$. 证明: $\mathscr{A}$ 最多有 $m+1$ 个不同的特征值.

Pf. Suppose there are $v_1, \cdots, v_{m+1}, v_{m+2} \in V$ and $\lambda_1, \cdots, \lambda_{m+2} \in K$ s.t.

$$\mathscr{A} v_i = \lambda_i v_i \quad \text{for } \lambda_i \neq \lambda_j \text{ if } i \neq j.$$

Thus, v_i's are linearly independent.

Since $\lambda_i v_i \in \operatorname{Im} \mathscr{A}$,

$$\dim \operatorname{Im} \mathscr{A} \geq \begin{cases} m+1, & \text{if } \exists i \text{ s.t. } \lambda_i = 0 \\ m+2, & \text{otherwise} \end{cases},$$

which contradicts with $\dim \operatorname{Im} \mathscr{A} = m$.

习题 5.0.27. 令 $V = K[X]_{\leq 4}$. 考虑 V 的有序基 $\mathcal{B} = (1,\ X-1,\ (X-1)^2,\ (X-1)^3,\ (X-1)^4)$.

1. 设 $\alpha = X^2 - 2 \in V$. 求 α 在有序基 $\mathcal{B}$ 下的坐标.

2. 定义线性变换

$$\mathscr{A} : V \longrightarrow V; \quad f(X) \mapsto X f'(X),$$

其中 $f'(X)$ 表示多项式 $f(X)$ 的形式导数. 求 $\mathscr{A}$ 在有序基 $\mathcal{B}$ 下的矩阵.

3. 求上个小题中线性变换 $\mathscr{A}$ 的特征多项式.

4. 令

$$U = \{f \in V \mid f(-X) = f(X)\},\ W = \{f \in V \mid f(-X) = -f(X)\}.$$

分别求 U 和 W 的一组基, 并证明 $V = U \oplus W$.

5. 证明: (4) 中的子空间 U 和 W 都是 (2) 中线性变换 $\mathscr{A}$ 的不变子空间.

6. 设 $\mathscr{A}$ 如问题 (2), U 和 W 如问题 (4). 求出 $\mathscr{A}|_U$ 和 $\mathscr{A}|_W$ 的特征多项式.

1) $\alpha(1) = -1$. $\alpha'(1) = 2X\big|_{X=1} = 2$. $\alpha''(1) = 2$. $\alpha'''(1) = \alpha^{(4)}(1) = 0$.

$\alpha = (-1, 2, 1, 0, 0)^T$

2) $\mathcal{A}(1)=0$, $\mathcal{A}(X-1) = X = X-1+1$, $\mathcal{A}((X-1)^2) = 2X(X-1) = 2(X-1)^2+2(X-1)$

$\mathcal{A}((X-1)^3) = 3X(X-1)^2 = 3(X-1)^3+3(X-1)^2$,

$\mathcal{A}((X-1)^4) = 4X(X-1)^3 = 4(X-1)^4+4(X-1)^3$, $\Rightarrow M_B(\mathcal{A}) = \begin{pmatrix} 0 & 1 & & & \\ 0 & 1 & 2 & & \\ 0 & & 2 & 3 & \\ 0 & & & 3 & 4 \\ 0 & & & & 4 \end{pmatrix}$

3) $|\lambda I - M_B(\mathcal{A})| = \lambda(\lambda-1)(\lambda-2)(\lambda-3)(\lambda-4)$

4) $1, X^2, X^4$ is a basis of $\mathcal{U}$. X, X^3 is a basis of W.

$\dim \mathcal{U} + \dim W = \dim V = 5$ and $\mathcal{U} \cap W = 0 \Rightarrow V = W \oplus \mathcal{U}$.

5) $\mathcal{A}(1) = 0 \in \mathcal{U}$, $\mathcal{A}(X^2) = 2X^2 \in \mathcal{U}$, $\mathcal{A}(X^4) = 4X^4 \in \mathcal{U}$. $\Rightarrow \mathcal{U}$ is $\mathcal{A}$-invariant.

$\mathcal{A}(X) = X \in W$, $\mathcal{A}(X^3) = 3X^3 \in W$. $\Rightarrow W$ is $\mathcal{A}$-invariant

6) $\mathcal{A}|_{\mathcal{U}} = \begin{pmatrix} 0 & & \\ & 2 & \\ & & 4 \end{pmatrix}$, $f_{\mathcal{U}} = \lambda(\lambda-2)(\lambda-4)$; $\mathcal{A}|_W = \begin{pmatrix} 1 & \\ & 3 \end{pmatrix}$, $f_W = (\lambda-1)(\lambda-3)$

Advanced Linear Algebra II - HW2

甘

习题 5.3.1. 对下列情况, 求出 f 和 g 的一个最大公因式 d 并写出一个 $d = uf + vg$ 形式的 Bézout 等式:

1. $f = X^4 + X^3 - 3X^2 - 4X - 1$, $g = X^3 + X^2 - X - 1$.
2. $f = X^4 + 2X^3 - X^2 - 4X - 2$, $g = X^4 + X^3 - X^2 - 2X - 2$.
3. $f = X^5 + 4X^4 + X^2 + 2X + 3$, $g = X - 2$.

1)

$$f - Xg = -2X^2 - 3X - 1$$

$$g + \frac{1}{2}(X - \frac{1}{2})(-2X^2 - 3X - 1) = -\frac{3}{4}(X+1)$$

$\Rightarrow \gcd(f, g) = X+1$ and $X+1 = -\frac{4}{3}g - \frac{2}{3}(X - \frac{1}{2})(f - Xg)$

$$= -\frac{2}{3}(X - \frac{1}{2})f + \frac{1}{3}(2X^2 - X - 4)g$$

3) $f - (X^4 + 6X^3 + 12X^2 + 25X + 52)g = 107$

$\Rightarrow d = 107$

习题 5.3.2. 设 $f, g \in K[X]$ 为非零多项式, $d = \gcd(f, g)$.

1. 证明: f/d 和 g/d 一定互素.
2. f/d 和 g 是否一定互素？请给出证明或反例.
3. $\gcd(f/d, g)$ 和 $\gcd(f, g/d)$ 二者是否有可能均不为 1？

1. If not. then $\exists h \in K[X]$ and $\deg h \geq 1$ s.t. $h \mid f/d$ and $h \mid g/d$

As a consequence, $hd \mid f$ and $hd \mid g$.

By the definition of gcd, $hd \mid d$, i.e., $h \mid 1$ which is a contradiction.

2. False. $f = x^2(x+1)$ $g = x(x+1)^2$, $d = \gcd(f, g) = x(x+1)$

$(f/d, g) = x$, $(f, g/d) = x+1$

3. Possible. The same example as in 2.

习题 5.3.3. 设 $f_1, \cdots, f_r \in K[X]$ 均为非零多项式, 其中 $r \geq 3$. 令 $f = \gcd(f_{r-1}, f_r)$. 对任意 $d \in K[X]$ 证明:

1. d 是 $f_1, \cdots, f_{r-2}, f_{r-1}, f_r$ 的公因式当且仅当 d 是 $f_1, \cdots, f_{r-2}, f$ 的公因式.
2. d 是 $f_1, \cdots, f_{r-2}, f_{r-1}, f_r$ 的最大公因式当且仅当 d 是 $f_1, \cdots, f_{r-2}, f$ 的最大公因式.

证: "⇒" $d \mid f_{r-1}$ 且 $d \mid f_r \underset{\text{推 5.3.6}}{\iff} d \mid \gcd(f_{r-1}, f_r) = f$

"⇐" 令 $D_1 := \{d$ 为 $f_1, \cdots, f_r$ 的公因式$\}$ $D_2 = \{d$ 为 $f_1, \cdots, f_{r-2}, f$ 的公因式$\}$

由 1. $D_1 = D_2$. 而最大公因式为各自集合的次数最高的因式

故等价.

习题 5.3.4. 设 $f_1, \cdots, f_r \in K[X]$ 不全为零, 其中 $r \geq 2$.

1. 证明: 多项式组 $f_1, \cdots, f_r$ 在 $K[X]$ 中一定有一个最大公因式 d 可以写成 $f_1, \cdots, f_r$ 的 $K[X]$-线性组合.
2. 假设 $d \in K[X]$ 是 $f_1, \cdots, f_r$ 的最大公因式, 并且 d 是 $f_1, \cdots, f_r$ 的 $K[X]$-线性组合. 证明:
   (a) 多项式组 $f_1, \cdots, f_r$ 在 $K[X]$ 中的任何公因式都是 d 的因式.
   (当然, d 的因式也都是 $f_1, \cdots, f_r$ 的公因式.)
   (b) 对于任意 $d' \in K[X]$, d' 也是 $f_1, \cdots, f_r$ 的最大公因式当且仅当 d' 是 d 的非零常数倍.
   (c) 多项式组 $f_1, \cdots, f_r$ 在 $K[X]$ 中的任意一个最大公因式都是 $f_1, \cdots, f_r$ 的 $K[X]$-线性组合.

根据以上结论可知, 存在唯一一个首一多项式是多项式组 $f_1, \cdots, f_r$ 在 $K[X]$ 中的最大公因式. 我们将这个最大公因式记为 $\gcd(f_1, \cdots, f_r)$ 或者 $\gcd(f_i)_{1 \leq i \leq r}$.

如果 $\gcd(f_1, \cdots, f_r) = 1$, 那么我们称 $f_1, \cdots, f_r$ 这个多项式组**互素**.

1. We prove by induction. $r=2$ is proved by Bezout identity.

Suppose it hold for $r-1$.

Let d be a GCD of $f_1, \cdots, f_r$, and d' be a GCD of $f_1, \cdots, f_{r-1}$.

Then $d \mid f_r$ and $d \mid d'$. By Kang's manuscript, d is $\gcd(f_r, d')$.

Thus, by Bezout identity, $\exists\, h_1 \,\&\, h_2 \in K[X]$ s.t. $h_1 f_r + h_2 d' = d$.

By induction hypothesis, d can be expressed as a $K[X]$-linear combination.

2. (a) Let d' be a common divisor of f_i. $\Rightarrow d' \mid f_i \ \forall i$

d is a $K[X]$-linear combination of f_i. $\Rightarrow d' \mid d$.

(b) If d' is another gcd. then $d' \mid d$ & $d \mid d'$ by (a)

Thus, $d' = kd$ for some $k \in K^*$.

(c) It follows immediately from 1 & 2(b)

习题 5.3.5. 设 $f_1, \cdots, f_r \in K[X]$ 均为非零多项式, 其中 $r \geq 2$.

1. 对每个 i, 令 $P_i := \prod_{1 \leq j \neq i \leq r} f_j$.
   证明: $f_1, \cdots, f_r$ 两两互素当且仅当对每个 $1 \leq i \leq r$, f_i 和 P_i 互素.
2. 证明: 如果 $f_1, \cdots, f_r$ 这些多项式两两互素, 那么多项式组 $f_1, \cdots, f_r$ (按习题 5.3.4 中的定义) 是互素的.
3. 举例说明: 当 $f_1, \cdots, f_r$ 构成的多项式组互素时, 这些多项式不一定两两互素.

Pf: 1. "$\Rightarrow$" Lemma 5.3.9.

"$\Leftarrow$" Suppose $\gcd(f_i, f_j) = d$ with $\deg d \geq 1$. and $i \neq j$.

Then $d \mid f_i$ & $d \mid P_i$, which is a contradiction.

2. If $\gcd(f_1, \cdots, f_r) = d \in K[X]$ with $\deg d \geq 1$.

Then $d \mid f_i$ for all i. Thus any two f_i & f_j are not coprime.

3. $f_1 = x(x-1)$, $f_2 = (x+1)x$ $f_3 = (x-1)(x+1)$

习题 5.3.6. 设 $f, g \in K[X]$ 均为首一 (故而非零) 多项式. 证明: $fg = \gcd(f,g)\mathrm{lcm}(f,g)$; 特别地, f 和 g 互素当且仅当 $\mathrm{lcm}(f,g) = fg$.

证 令 $\gcd(f,g) = d$, $\mathrm{lcm}(f,g) = D$, 则只需证 $D = \frac{fg}{d}$.

首先, $\frac{fg}{d}$ 显然为 f 与 g 的倍式. 即 $D \mid \frac{fg}{d}$. (思考题 5.22)

设 $\frac{fg}{d} = D \cdot h$. $h \in K[X] \setminus \{0\}$. 则 $h \cdot \frac{D}{f} = \frac{g}{d}$ 且 $h \cdot \frac{D}{g} = \frac{f}{d}$.

于是 $h \mid \frac{g}{d}$ & $\frac{f}{d}$. 由于 $(\frac{f}{d}, \frac{g}{d}) = 1$, $h \in K \setminus \{0\}$.

再由首项系数知 $h = 1$. 故 $D = f \cdot g / d$

习题 5.3.7. 设 $f_1, \cdots, f_r \in K[X]$ 均为首一多项式, 其中 $r \geq 2$.

1. 证明: $f_1, \cdots, f_r$ 这些多项式两两互素当且仅当 $\mathrm{lcm}(f_1, \cdots, f_r) = f_1 \cdots f_r$.
   (提示: 如果不使用 (5.3.14) 一段所提到的多项式不可约因式分解理论, 此结论的证明可以通过对 r 的归纳法来完成.)
2. 举例说明: 当 $r \geq 3$ 时, 等式 $f_1 \cdots f_r = \gcd(f_1, \cdots, f_r)\mathrm{lcm}(f_1, \cdots, f_r)$ 不一定成立.

Pf: We prove by induction on r. When $r=2$, we have proved in 5.3.6.

If $f_1, \cdots, f_r$ are pairwise coprime, in particular $f_1, \cdots, f_{r-1}$ are pairwise coprime, by induction hypo, $\mathrm{lcm}(f_1, \cdots, f_{r-1}) = f_1 \cdots f_{r-1}$.

Moreover f_r and $f_1 f_2 \cdots f_{r-1}$ are coprime.

Thus, $\mathrm{lcm}(f_1, \cdots, f_r) = \mathrm{lcm}(f_r, f_1 f_2 \cdots f_{r-1}) = f_1 f_2 \cdots f_r$.

Conversely, suppose $d = \gcd(f_i, f_j)$ with $\deg d \geq 1$ & $i \neq j$.

Then $f_1 \cdots f_r / d$ is a common multiple of f_i, which is a contradiction with the definition of lcm.

2. Let $f_1 = x$, $f_2 = (x+1)$, $f_3 = x(x+1)$.

Then $\gcd(f_1, f_2, f_3) = 1$. $\mathrm{lcm}(f_1, f_2, f_3) = f_3$. But $f_3 \neq f_1 f_2 f_3$.

习题 5.3.8 Let $g, h \in K[X]$. If $P \in K[X]$ and g are coprime. and $P \mid gh$.
Prove $P \mid h$.

Pf. By Bezout's identity, $\exists\, a, b \in K[X]$ s.t. $aP + bg = 1$.

$\Rightarrow aPh + bgh = h$. Since $P \mid aPh + bgh$, then $P \mid h$.

Advanced Linear Algebra II - HW3

$\boxed{2}$

习题 5.1.1. 定义复向量空间 $\mathbb{C}^3$ 上的线性变换 $\mathscr{A}$ 如下:

$$\mathscr{A}:\ \mathbb{C}^3 \longrightarrow \mathbb{C}^3\ ; \qquad \begin{pmatrix} x \\ y \\ z \end{pmatrix} \longmapsto \begin{pmatrix} 3x+y-2z \\ -x+5z \\ -x-y+4z \end{pmatrix}.$$

求 $\mathscr{A}$ 的所有特征值及相应的广义特征子空间.

Pf. Let $e_i = (0,\cdots,\underset{\substack{\uparrow \\ i}}{1},\cdots 0)^T$. Then $\mathcal{E} = (e_1, e_2, e_3)$ is an ordered basis.

$$M_{\mathcal{E}}(\mathscr{A}) = \begin{pmatrix} 3 & 1 & -2 \\ -1 & 0 & 5 \\ -1 & -1 & 4 \end{pmatrix}$$

$$|\lambda I_3 - M_{\mathcal{E}}(\mathscr{A})| = \begin{vmatrix} \lambda-3 & -1 & 2 \\ 1 & \lambda & -5 \\ 1 & 1 & \lambda-4 \end{vmatrix} = \begin{vmatrix} \lambda-2 & 0 & \lambda-2 \\ 0 & \lambda-1 & -\lambda-1 \\ 1 & 1 & \lambda-4 \end{vmatrix}$$

$$= (\lambda-2)(\lambda-1)(\lambda-4) - (\lambda-1)(\lambda-2) + (\lambda+1)(\lambda-2)$$

$$= (\lambda-2)(\lambda^2-5\lambda+6) = (\lambda-2)^2(\lambda-3)$$

Then $\lambda_1 = 2$ & $\lambda_2 = 3$ are all eigenvalues of $\mathscr{A}$.

① $$2I_3 - M_{\mathcal{E}}(\mathscr{A}) = \begin{pmatrix} -1 & -1 & 2 \\ 1 & 2 & -5 \\ 1 & 1 & -2 \end{pmatrix}$$

$$\left(2I_3 - M_{\mathcal{E}}(\mathscr{A})\right)^2 = \begin{pmatrix} 2 & 1 & -1 \\ -4 & -2 & 2 \\ -2 & -1 & 1 \end{pmatrix}$$

$$\left(2I_3 - M_{\mathcal{E}}(\mathscr{A})\right)^2 X = 0 \iff 2x_1 + x_2 - x_3 = 0 \iff X = a_1\begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix} + a_2\begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} \text{ for any } a_i \in \mathbb{K}.$$

$$G(2,\mathscr{A}) = \ker(2I-\mathscr{A})^2 = \operatorname{span}_{\mathbb{K}}\left\{\begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}\right\}$$

② $3I_3 - M_{\mathcal{E}}(\mathscr{A}) = \begin{pmatrix} 0 & -1 & 2 \\ 1 & 3 & -5 \\ 1 & 1 & -1 \end{pmatrix}.$

$\left(3I_3 - M_{\mathcal{E}}(\mathscr{A})\right) X = 0 \Leftrightarrow X \in \mathbb{K}(-1, 2, 1)^T$

$\Rightarrow G(3, \mathscr{A}) = \mathbb{K}(-1, 2, 1)^T$

习题 5.1.2. 设 $\mathscr{A}$ 是 K-向量空间 V 上的可逆线性变换, $\mathscr{A}^{-1}$ 为其逆变换. 证明: 对于任意 $0 \neq \lambda \in K$, $G(\lambda, \mathscr{A}) = G(\lambda^{-1}, \mathscr{A}^{-1})$.

Pf. $G(\lambda, \mathscr{A}) = \{ v \in V : (\lambda I - \mathscr{A})^k v = 0 \text{ for some } k \in \mathbb{Z}_+ \}$

For any $v \in G(\lambda, \mathscr{A})$, assume $k \in \mathbb{Z}_+$ s.t. $(\lambda I - \mathscr{A})^k v = 0$.

Then $(\lambda^{-1} I - \mathscr{A}^{-1})^k v = \mathscr{A}^{-k} (\lambda^{-1} \mathscr{A} - I)^k v$

$= \lambda^{-k} \mathscr{A}^{-k} (\mathscr{A} - \lambda I)^k v = 0$

$\Rightarrow v \in G(\lambda^{-1}, \mathscr{A}^{-1})$ vice versa.

习题 5.1.3. 举例说明存在实方阵 A 满足: A 不是幂零矩阵, 但 0 是 A 唯一的实特征值.

$A = \begin{pmatrix} 0 & & \\ & 0 & -1 \\ & 1 & 0 \end{pmatrix}$, $|\lambda I_3 - A| = \lambda(\lambda^2 + 1) = 0 \Rightarrow \lambda = 0, \pm i$.

$A^2 = \begin{pmatrix} 0 & & \\ & -1 & \\ & & -1 \end{pmatrix} \Rightarrow \operatorname{rank} A^n = 2$. for any $n \geq 1$.

习题 5.1.5. 设 $\mathscr{A}, \mathscr{B} \in \mathrm{End}(V)$. 假设 $\mathscr{A}, \mathscr{B}$ 都是幂零变换.

1. 证明: 若 $\mathscr{A}, \mathscr{B}$ 可交换, 则 $\mathscr{A} + \mathscr{B}$ 一定是幂零变换.
2. 举例说明: 若 $\mathscr{A}, \mathscr{B}$ 不可交换, 则 $\mathscr{A} + \mathscr{B}$ 可能不是幂零变换.
3. $\mathscr{A}\mathscr{B}$ 是否一定是幂零变换? 若是, 请给出证明; 若否, 请举出反例.

证: 1. 设 $\mathscr{A}^n = \mathscr{B}^m = 0$, $n, m \in \mathbb{Z}_+$, 则 $(\mathscr{A} + \mathscr{B})^{n+m-1} = \sum_{i=0}^{m+n-1} \binom{n+m-1}{i} \mathscr{A}^{n+m-1-i} \mathscr{B}^i = 0$

2. 令 $\dim V = 2$, $M_{\mathcal{E}}(\mathscr{A}) = \begin{pmatrix} & 0 \\ 1 & \end{pmatrix}$, $M_{\mathcal{E}}(\mathscr{B}) = \begin{pmatrix} & 1 \\ 0 & \end{pmatrix}$

$M_\varepsilon(\mathscr{A}+\mathscr{B}) = \begin{pmatrix} 1 & 1 \end{pmatrix}$，$\operatorname{rank}\begin{pmatrix} 1 & 1 \end{pmatrix} = 2$. 故 $\mathscr{A}+\mathscr{B}$ 不幂零

3. $\mathscr{A}$，$\mathscr{B}$ 取 2 中变换. $M_\varepsilon(\mathscr{A}\mathscr{B}) = \begin{pmatrix} 0 & \\ & 1 \end{pmatrix}$，故 $\mathscr{A}\mathscr{B}$ 不幂零.

习题 5.1.6. 设 $\mathscr{A} \in \mathrm{End}(V)$ 为幂零变换. 对任意非零常数 $\alpha \in K$, 证明 $\alpha I + \mathscr{A}$ 是可逆变换并求出 $(\alpha I + \mathscr{A})^{-1}$.

$$\alpha\left(I + \frac{1}{\alpha}\mathscr{A}\right)\left(I - \frac{\mathscr{A}}{\alpha} + \frac{\mathscr{A}^2}{\alpha^2} - \frac{\mathscr{A}^3}{\alpha^3} + \cdots + (-1)^n\frac{\mathscr{A}^n}{\alpha^n} + \cdots\right) = \alpha$$

$\Rightarrow (\alpha I + \mathscr{A})^{-1} = \frac{I}{\alpha} - \frac{\mathscr{A}}{\alpha^2} + \frac{\mathscr{A}^2}{\alpha^3} + \cdots + (-1)^n\frac{\mathscr{A}^n}{\alpha^{n+1}} + \cdots$ makes sense because $\mathscr{A}$ is nilpotent.

requires $\mathbb{K}$ is algebraically closed.

习题 5.1.9. 设 $n = \dim V$, $\mathscr{A} \in \mathrm{End}(V)$ 为幂零变换. 设 $\dim E(0, \mathscr{A}) = k$. 证明 $\mathscr{A}^{n-k+1} = 0$.

Pf. <u>Method 1</u>: Let $J = \begin{pmatrix} J_1 & & \\ & \ddots & \\ & & J_k \end{pmatrix}$ be the corresponding Jordan matrix of $\mathscr{A}$, where $J_i = \begin{pmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & 0 \end{pmatrix}$ is a $n_i \times n_i$ matrix.

WLOG, we assume $n_1 \leq n_2 \leq \cdots \leq n_k$.

Then $J^{n_k} = 0$ and $n_k = n - n_1 - n_2 - \cdots - n_{k-1} \leq n-(k-1)$.

Thus $\mathscr{A}^{n-(k-1)} = 0$.

<u>Method 2</u>: Use the result of Ex 5.1.7.

By the conditions, we have $\ker(\mathscr{A}-0) = k$. Then by Ex 5.1.7

$\dim\ker(\mathscr{A}^i) + 1 \leq \dim\ker(\mathscr{A}^{i+1})$ or $\dim\ker(\mathscr{A}^{i+1}) = \dim V = n$.

$\Rightarrow \dim\ker(\mathscr{A}^{i+1}) \geq \min\{n, \dim\ker(\mathscr{A}^i) + 1\}$

Iterating, we get $\dim\ker(\mathscr{A}^{n-k+1}) \geq \min\{n, \dim\ker\mathscr{A} + n - k\}$

$= n$

$\Rightarrow \dim\ker\mathscr{A}^{n-k+1} = n \Leftrightarrow \mathscr{A}^{n-k+1} = 0$

习题 5.1.12. 对任意 $\mathscr{A} \in \mathrm{End}(V)$ 和任意正整数 m 证明: $\dim \mathrm{Ker}(\mathscr{A}^m) \le m \dim \mathrm{Ker}(\mathscr{A})$. (提示: 考虑映射 $\mathrm{Ker}(\mathscr{A}^{m+1}) \to \mathrm{Ker}(\mathscr{A}^m)$; $x \mapsto \mathscr{A} x$.)

Pf. Consider $\ker(\mathscr{A}^{m+1}) \xrightarrow{\varphi_m} \ker \mathscr{A}^m$; $x \mapsto \mathscr{A}x$

Let $x \in \ker(\mathscr{A}^{m+1})$, $\mathscr{A}^m \varphi_m(x) = \mathscr{A}^{m+1}(x) = 0 \Rightarrow \varphi_m$ well-defined

$\Rightarrow \dim \ker \mathscr{A}^{m+1} = \dim \mathrm{Im}\, \varphi_m + \dim \ker \varphi_m$

$= \dim \ker \mathscr{A}^m + \dim \ker \varphi_m$

Thus, $\dim \mathrm{Ker}\, \mathscr{A}^m = \dim \ker \varphi_{m-1} + \dim \ker \varphi_{m-2} + \cdots + \dim \ker \varphi_1 + \dim \ker \mathscr{A}$

Note that for any φ_i, $\ker \varphi_i \subset \ker \mathscr{A}$:

$\forall x \in \ker \varphi_i$, i.e., $x \in \mathrm{Ker}\, \varphi_{i+1}$ and $\mathscr{A}x = 0$, then $x \in \mathrm{Ker}\, \mathscr{A}$

Therefore, $\dim \mathrm{Ker}\, \mathscr{A}^m \le m \dim \ker \mathscr{A}$.

习题 5.1.13. 设 $n = \dim V$, $\mathscr{A} \in \mathrm{End}(V)$. 假设 $\ker(\mathscr{A}^n) \ne \ker(\mathscr{A}^{n-1})$. 证明 $\mathscr{A}$ 是幂零变换并对每个 $j \in \mathbb{N}$ 求出 $\dim \ker(\mathscr{A}^j)$.

Pf. Note that $\ker(\mathscr{A}) \subset \ker \mathscr{A}^2 \subset \cdots \subset \ker \mathscr{A}^{n-1} \subsetneq \ker \mathscr{A}^n \subset V$

Claim: $\forall 1 \le i \le n-1$, $\ker \mathscr{A}^i \subsetneq \ker \mathscr{A}^{i+1}$.

Suppose $\exists 1 \le i \le n-1$ s.t. $\ker \mathscr{A}^i = \ker \mathscr{A}^{i+1}$. Obviously, $i \ne n-1$.

$\forall v \in \ker \mathscr{A}^n$, $\mathscr{A}^{n-i-1} v \in \ker \mathscr{A}^{i+1} = \ker \mathscr{A}^i$

$\Rightarrow \mathscr{A}^i \mathscr{A}^{n-i-1} v = \mathscr{A}^{n-1} v = 0 \Rightarrow v \in \ker \mathscr{A}^{n-1}$

$\Rightarrow \ker \mathscr{A}^n \subseteq \ker \mathscr{A}^{n-1}$, which is a contradiction.

Since $\ker \mathscr{A}^i$'s are subspaces,

$0 \leq \dim \ker \mathscr{A} < \dim \ker \mathscr{A}^2 < \cdots < \dim \ker \mathscr{A}^{n-1} < \dim \ker \mathscr{A}^n \leq n$

Thus, $\dim \ker \mathscr{A}^i = i$ for any $1 \leq i \leq n$ and

$\dim \ker \mathscr{A}^{n+i} = n$ for all $n \in \mathbb{N}$

习题 5.1.14. 设 $n = \dim V \geq 2$, $\mathscr{A} \in \mathrm{End}(V)$ 满足 $\ker(\mathscr{A}^{n-2}) \neq \ker(\mathscr{A}^{n-1})$. 证明: $\mathscr{A}$ 最多有两个不同的特征值.

Pf. By a similar argument as Ex 5.1.13,

$\dim \ker(\mathscr{A}^{n-1}) = n-1 = \dim G(0, \mathscr{A})$

Since V can be decomposed as a direct sum of $G(\lambda, \mathscr{A})$'s,

$$\dim \bigoplus_{\lambda \neq 0} G(\lambda, \mathscr{A}) = n-(n-1) = 1$$

$\Rightarrow$ At most 2 eigenvalues.

requires $\mathbb{K}$ is alg. closed

Suppose there are more than two distinct eigenvalues, then $\dim V \geq n+1$, which is a contradiction!

习题 5.1.15. 设 $n = \dim V \geq 2$, $\mathscr{A} \in \mathrm{End}(V)$. 假设 K 中有两个不同的非零常数都是 $\mathscr{A}$ 的特征值. 证明:

1. 对任意自然数 $m \geq n-2$ 均有 $\mathrm{Ker}(\mathscr{A}^m) = \mathrm{Ker}(\mathscr{A}^{n-2})$.
2. $V = \mathrm{Ker}(\mathscr{A}^{n-2}) \oplus \mathrm{Im}(\mathscr{A}^{n-2})$.

Pf. Suppose $\mathscr{A}$ has two nonzero eigenvalues λ_1, λ_2.

$$V \supseteq \bigoplus_{\lambda \in \{0, \lambda_1, \lambda_2\}} G(\lambda, \mathscr{A}) \supseteq \bigoplus_{\lambda \in \{\lambda_1, \lambda_2\}} \ker(\mathscr{A} - \lambda I) \oplus G(0, \mathscr{A})$$

$\Rightarrow \dim G(0,\mathscr{A}) \leq n - \dim \ker(\mathscr{A}-\lambda_1 I) - \dim \ker(\mathscr{A}-\lambda_2 I) \leq n-2$

因此 $\ker \mathscr{A}^{n-2} = G(0,\mathscr{A}) = \ker \mathscr{A}^n$

故 $\ker \mathscr{A}^m = \ker \mathscr{A}^{n-2} \quad \forall m \geq n-2$

2\. $\forall v \in \ker \mathscr{A}^{n-2} \cap \operatorname{Im} \mathscr{A}^{n-2}$, $\exists v' \in V$ s.t. $\mathscr{A}^{n-2} v' = v$ 且 $\mathscr{A}^{n-2} v = 0$.

$\forall n \geq 2$, $0 = \mathscr{A}^{n-2} v = \mathscr{A}^{2n-4} v' = \mathscr{A}^{n-2} v' = v$. 由维度公式, $V = \ker \mathscr{A}^{n-2} \oplus \operatorname{Im} \mathscr{A}^{n-2}$.

习题 5.1.16. 设 A 为 $\mathbf{M}_n(K)$ 中的幂零矩阵. 定义 $V = \mathbf{M}_n(K)$ 上的线性变换

$$\mathscr{A} : V \longrightarrow V ; \quad X \longmapsto AX - XA.$$

证明 $\mathscr{A}$ 是个幂零变换.

Pf

$$\mathscr{A}^m X = \sum_{i=0}^{m} (-1)^i \binom{m}{i} A^{m-i} X A^i$$

If $A^k = 0$, then $\mathscr{A}^{2k} X = \sum (-1)^i \binom{2k}{i} A^{2k-i} X A^i = 0$

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习题 5.1.17. 假设 $\dim V = 3$, $\mathscr{A} \in \mathrm{End}(V)$, $\mathcal{B} = (\varepsilon_1, \varepsilon_2, \varepsilon_3)$ 为 V 的一组有序基. 令 $A = \mathcal{M}_{\mathcal{B}}(\mathscr{A})$. 对以下两种情况分别验证 $\mathscr{A}$ 是幂零变换并求出 $\mathscr{A}$ 的一组 Jordan 基:

$$(1)\ A = \begin{pmatrix} 0 & -3 & 3 \\ -2 & -7 & 13 \\ -1 & -4 & 7 \end{pmatrix}; \qquad (2)\ A = \begin{pmatrix} 3 & 6 & -15 \\ 1 & 2 & -5 \\ 1 & 2 & -5 \end{pmatrix}.$$

$$A \to \begin{pmatrix} 0 & -3 & 3 \\ -1 & 0 & 3 \\ 0 & -4 & 4 \end{pmatrix} \Rightarrow \text{rank } A = 2$$

$$A^2 = \begin{pmatrix} 3 & 9 & -18 \\ 1 & 3 & -6 \\ 1 & 3 & -6 \end{pmatrix} \Rightarrow \text{rank } A = 1.$$

$$A^3 = 0 \quad \Rightarrow V = \ker \mathscr{A}^3 = G(0, \mathscr{A})$$

$$\begin{pmatrix} 3 & 9 & -18 \\ 1 & 3 & -6 \\ 1 & 3 & -6 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$\Rightarrow$ The Jordan form of A is $\begin{pmatrix} 0 & 1 & \\ & 0 & 1 \\ & & 0 \end{pmatrix}$

Let (v_1, v_2, v_3) be the corresponding Jordan basis.

Then $v_1 \in \mathrm{Im}(\mathscr{A}^2) = \mathbb{K}\, \mathcal{B}\begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix} = \mathbb{K}(3e_1 + e_2 + e_3)$

Take $v_1 = 3e_1 + e_2 + e_3$, then $\mathscr{A}^2 v_3 = v_1$.

Take $v_3 = e_1$. Then

$v_1 = \mathscr{A}^2 v_3 = 3e_1 + e_2 + e_3$, $v_2 = \mathscr{A} v_3 = -2e_2 - e_3$, $v_3 = e_1$ is a Jordan basis

习题 5.1.18. 设 $A \in \mathbf{M}_n(K)$, $V = \mathbf{M}_n(K)$. 定义线性变换 $\mathscr{A} \in \mathrm{End}(V)$ 为

$$\forall X \in V = \mathbf{M}_n(K)\,, \quad \mathscr{A}(X) := A^T X A\,.$$

1. 证明: 若 A 是幂零矩阵, 则 $\mathscr{A}$ 是幂零变换.
2. 假设 $n = 2$, $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. 找出 $\mathscr{A}$ 在 V 中的一组 Jordan 基.
3. 证明: 若 $\mathscr{A}$ 是幂零变换, 则 A 是幂零矩阵.

1. Suppose $A^m = 0$, $\mathscr{A}^m(X) = (A^T)^m X (A^m) = 0$ for any $X \in V$.

2. Take $\mathcal{B} = (E_{11}, E_{12}, E_{21}, E_{22})$ as a basis of V.

$$M_{\mathcal{B}}(\mathscr{A}) = \begin{pmatrix} & 0 \\ 1 & \end{pmatrix} \Rightarrow \operatorname{Im}\mathscr{A} = \mathbb{K} E_{22} \text{ and } \mathscr{A}E_{11} = E_{22}$$

$$\ker \mathscr{A} = \operatorname{span}\{E_{12}, E_{21}, E_{22}\}$$

$\Rightarrow \exists$ Jordan basis $\mathcal{C} = (w_1, w_2, w_3, w_4)$ s.t. $M_{\mathcal{C}}(\mathscr{A}) = \begin{pmatrix} 0 & & & \\ & 0 & & \\ & & 0 & 1 \\ & & & 0 \end{pmatrix}$

Take $w_4 = E_{11}$ and $w_3 = E_{22}$. $w_1 = E_{12}$, $w_2 = E_{21}$. which satisfy

$$\mathscr{A}(w_1, w_2, w_3, w_4) = (0, 0, 0, w_3)$$

3. Suppose $\mathscr{A}^m = 0$, then for all $X \in M_n(\mathbb{K})$, $A^{Tm} X A^m = 0$

Denote $A^m = (a_{ij})$ and take $X = E_{ii}$. We get

$$A^{Tm} X A^m = \sum_{p,q} a_{ip}\, a_{iq}\, E_{pq} = 0$$

In particular, for any j, $a_{ij}^2 = 0 \Rightarrow a_{ij} = 0$ for any i, j

$\Rightarrow A^m = 0$

习题 5.1.20. 验证以下矩阵 A 为幂零矩阵, 并将其化为 Jordan 标准形:

$$(1)\ \ A=\begin{pmatrix}0&2&3&4\\0&0&2&3\\0&0&0&2\\0&0&0&0\end{pmatrix};\quad (2)\ \ A=\begin{pmatrix}2&1&0&0\\-4&-2&0&0\\7&1&1&1\\-17&-6&-1&-1\end{pmatrix};\quad (3)\ \ A=\begin{pmatrix}0&3&0&-3\\2&7&0&-13\\0&3&0&-3\\1&4&0&-7\end{pmatrix}.$$

(1). $A^2=\begin{pmatrix}0&0&4&12\\ &0&0&4\\ & &0&0\\ & & &0\end{pmatrix}$ $\qquad A^3=\begin{pmatrix} & & &8\\ & & & \\ & & & \\ & & & \end{pmatrix}$ $\qquad A^4=0$

The Jordan form of A is $\begin{pmatrix}0&1& & \\ &0&1& \\ & &0&1\\ & & &0\end{pmatrix}$ Goal: Find $(\xi_1,\cdots\xi_4)$ s.t. $A(\xi_1,\cdots\xi_4)=(\xi_1,\cdots\xi_4)J$

Note that $A^3\begin{pmatrix}0\\0\\0\\1\end{pmatrix}=\begin{pmatrix}8\\0\\0\\0\end{pmatrix}\neq 0$

Take $\xi_4=\begin{pmatrix}0\\0\\0\\1\end{pmatrix}$, $\xi_3=A\xi_4=\begin{pmatrix}4\\3\\2\\0\end{pmatrix}$, $\xi_2=A^2\xi_4=\begin{pmatrix}12\\4\\0\\0\end{pmatrix}$, $\xi_1=A^3\xi_4=\begin{pmatrix}8\\0\\0\\0\end{pmatrix}$

Then

$$A(\xi_1,\xi_2,\xi_3,\xi_4)=(0,\xi_1,\xi_2,\xi_3)$$
$$=(\xi_1\ \xi_2\ \xi_3\ \xi_4)\begin{bmatrix}0&1&0&0\\0&0&1&0\\0&0&0&1\\0&0&0&0\end{bmatrix}$$

$$\Rightarrow\ J_4(0)=\begin{bmatrix}8&12&4& \\ &4&3& \\ & &2& \\ & & &1\end{bmatrix}^{-1}A\begin{bmatrix}8&12&4& \\ &4&3& \\ & &2& \\ & & &1\end{bmatrix}$$

$$(2)\ \ A=\begin{pmatrix}2&1&0&0\\-4&-2&0&0\\7&1&1&1\\-17&-6&-1&-1\end{pmatrix};$$

$$A^2=\begin{vmatrix}0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{vmatrix}$$

$\#$ Jordan blocks $=\dim \underset{\substack{\uparrow\\ \text{null space of } A}}{N(A)}=4-\operatorname{rank}A=2$

Thus the Jordan form is $\begin{pmatrix}0&1& & \\ &0& & \\ & &0&1\\ & & &0\end{pmatrix}$ Goal: Find $\underset{\mathcal{E}}{\underbrace{(\xi_1,\xi_2,\xi_3,\xi_4)}}$ s.t. $A\mathcal{E}=\mathcal{E}J$

Take $\xi_2=(1,0,0,0)^T$, $\xi_4=(0,0,1,0)^T$, $\xi_1=A\xi_2=\begin{pmatrix}2\\-4\\7\\-17\end{pmatrix}$, $\xi_3=A\xi_4=\begin{pmatrix}0\\0\\1\\-1\end{pmatrix}$

Then

$$A(\xi_1,\xi_2,\xi_3,\xi_4) = (0,\xi_1,0,\xi_3)$$

$$= (\xi_1\ \xi_2\ \xi_3\ \xi_4)\begin{bmatrix} J_2(0) & \\ & J_2(0)\end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} J_2(0) & \\ & J_2(0)\end{bmatrix} = \begin{pmatrix} 2 & 1 & 0 & 0 \\ -4 & 0 & 0 & 0 \\ 7 & 0 & 1 & 1 \\ -17 & 0 & -1 & 0 \end{pmatrix}^{-1} A \begin{pmatrix} 2 & 1 & 0 & 0 \\ -4 & 0 & 0 & 0 \\ 7 & 0 & 1 & 1 \\ -17 & 0 & -1 & 0 \end{pmatrix}$$

习题 5.1.21. 设 $A \in \mathbf{M}_2(K)$. 假设存在矩阵 $B \in \mathbf{M}_2(K)$ 使得 $AB - BA = A$.

1. 证明 $\mathrm{Tr}(A) = \mathrm{Tr}(A^2) = 0$.
2. 证明 A 是幂零矩阵.

$$(A-\lambda)B\xi = A\xi = \lambda\xi$$

1. $\mathrm{Tr}(A) = \mathrm{Tr}(AB-BA) = \mathrm{Tr}(AB) - \mathrm{Tr}(BA)$

$$= \mathrm{Tr}(AB) - \mathrm{Tr}(AB) = 0$$

$$\mathrm{Tr}(A^2) = \mathrm{Tr}(AB\cdot A - BA\cdot A)$$

$$= \mathrm{Tr}(ABA) - \mathrm{Tr}(ABA)$$

$$= 0$$

2. Suppose A is NOT nilpotent.

We consider $A \in M_2(\mathbb{K}) \subseteq M_2(\mathbb{C})$ and $\mathcal{A}: \mathbb{C}^2 \to \mathbb{C}^2$ by

$$\mathcal{A}(X) = AX \quad \text{for all } X \in \mathbb{C}^2$$

Then $\mathcal{A}$ has a nonzero eigenvalue. (Otherwise, $\mathbb{C}^2 = G(0,\mathcal{A})$, then

$\mathrm{Tr}\, J =$

$\mathcal{A}$ is nilpotent $\Rightarrow A = M_\varepsilon(\mathcal{A})$ nilpotent, where $\varepsilon = \left(\binom{1}{0}, \binom{0}{1}\right)$.)

Since $\mathrm{Tr}\, A = 0$, A must have two nonzero eigenvalues $\lambda, -\lambda$.

Then the Jordan form of A is $J = \begin{pmatrix} \lambda & \\ & -\lambda \end{pmatrix}$. $\lambda \neq 0$

Thus, the Jordan form of A^2 is $J^2 = \begin{pmatrix} \lambda^2 & \\ & \lambda^2 \end{pmatrix}$.

It implies $\mathrm{Tr}(A^2) = \mathrm{Tr}(J^2) = 2\lambda^2 = 0$, i.e., $\lambda = 0$.

Contradiction!

习题 5.1.22. 设 $\mathscr{A} \in \mathrm{End}(V)$ 在 V 的一组有序基 $\varepsilon_1, \cdots, \varepsilon_n$ 下的矩阵是

$$A = \begin{pmatrix} 0 & & & & \\ -1 & 0 & & & \\ & -1 & 0 & & \\ & & \ddots & \ddots & \\ & & & -1 & 0 \end{pmatrix}.$$

求 $\mathscr{A}$ 在 V 中的一组 Jordan 基.

The Jordan canonical form of A is $\begin{pmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & 0 \end{pmatrix} = J_n(0)$.

Then $(\varepsilon_n, -\varepsilon_{n-1}, \varepsilon_{n-2}, \cdots, (-1)^{n-1}\varepsilon_1)$ is a Jordan basis of $\mathscr{A}$.

习题 5.2.1. 将以下矩阵化为 Jordan 标准形:

$$(1) \quad \begin{pmatrix} 0 & 1 & 0 \\ -4 & 4 & 0 \\ -2 & 1 & 2 \end{pmatrix};$$

← Denote by A

$$|\lambda I_3 - A| = \begin{vmatrix} \lambda & -1 & 0 \\ 4 & \lambda-4 & 0 \\ 2 & -1 & \lambda-2 \end{vmatrix} = (\lambda-2)^3$$

$$2I_3 - A = \begin{pmatrix} 2 & -1 & 0 \\ 4 & -2 & 0 \\ 2 & -1 & 0 \end{pmatrix} \Rightarrow \dim N(A) = 3 - \mathrm{rank}\, A = 2$$

The Jordan canonical form $J = \begin{pmatrix} 2 & & \\ & 2 & 1 \\ & & 2 \end{pmatrix}$.

Note that $N(A) = \mathrm{span}_K \left\{ \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} \right\}$. Moreover,

null space of A

$$(2I_3 - A)\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 \\ -2 \\ -1 \end{pmatrix} \in N(A).$$

Take $\xi_3 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$, $\xi_2 = (2I_3 - A)\xi_3 = \begin{pmatrix} -1 \\ -2 \\ -1 \end{pmatrix}$, $\xi_1 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

Then

$$A(\xi_1, \xi_2, \xi_3) = (2\xi_1, 2\xi_2, \xi_2 + 2\xi_3)$$

$$= (\xi_1 \xi_2 \xi_3) J$$

$$\Rightarrow J = \begin{pmatrix} & -1 & \\ & -2 & 1 \\ 1 & -1 & \end{pmatrix}^{-1} A \begin{pmatrix} & -1 & \\ & -2 & 1 \\ 1 & -1 & \end{pmatrix}$$

习题 5.2.2. 将以下矩阵化为 Jordan 标准形:

$$(1)\ \begin{pmatrix} 2 & 2 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & -1 & -1 & 1 \\ 0 & 0 & 0 & -1 & 2 \\ 0 & 0 & 0 & 0 & -1 \end{pmatrix};\quad (2)\ \begin{pmatrix} 2 & 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 & 0 & 0 \\ -1 & 0 & 2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 2 & 0 & 0 \\ 1 & 1 & 1 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 \end{pmatrix}.$$

Denote by A.

$|\lambda I_6 - A| = (\lambda - 2)^5 (\lambda + 1)$

方便记号

Denote by $\mathcal{A}: \mathbb{K}^6 \to \mathbb{K}^6;\ X \mapsto AX$.

① $\lambda = -1$. $\ker(-\mathrm{Id} - \mathcal{A}) = \left\{ X \in \mathbb{K}^6 : \begin{bmatrix} -3 & & & & & \\ -1 & -3 & & & & \\ 1 & & -3 & & & \\ & -1 & & -3 & & \\ -1 & -1 & -1 & -1 & -3 & \\ & & & & -1 & 0 \end{bmatrix} X = 0 \right\}$

$= \mathbb{K}(0,0,0,0,0,1)^T$

② $\lambda = 2$,

$$2I_6 - A = \begin{bmatrix} 0 & & & & & \\ -1 & 0 & & & & \\ 1 & & 0 & & & \\ & -1 & & 0 & & \\ -1 & -1 & -1 & -1 & 0 & \\ & & & & -1 & 3 \end{bmatrix},\ \mathrm{rank}(2I_6 - A) = 4$$

2 blocks of $\lambda = 2$

1 block has size ≥ 2

$$(2I_6 - A)^2 = \begin{bmatrix} 0 & & & & & \\ 0 & 0 & & & & \\ 0 & 0 & 0 & & & \\ 1 & 0 & 0 & 0 & & \\ 0 & 1 & 0 & 0 & 0 & \\ 1 & 1 & 1 & 1 & -3 & 9 \end{bmatrix},\ \mathrm{rank}(2I_6 - A)^2 = 3$$

$$(2I_6 - A)^3 = \begin{bmatrix} & & & O & & \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 3 & 2 & 3 & 3 & -9 & 27 \end{bmatrix}, \quad \mathrm{rank}(2I_6 - A)^3 = 2$$

$\nearrow$ 1 block has size ≥ 3

$\searrow$ 1 block has size ≥ 4

$\nearrow$

$$(2I_6 - A)^4 = \begin{bmatrix} & & & O & & \\ 10 & 6 & 9 & 9 & -27 & 81 \end{bmatrix} \quad \mathrm{rank}(2I_6 - A)^4 = 1$$

$\Rightarrow$ Jordan form of A is $J = \begin{bmatrix} -1 & & & & & \\ & 2 & & & & \\ & & 2 & 1 & & \\ & & & 2 & 1 & \\ & & & & 2 & 1 \\ & & & & & 2 \end{bmatrix}$

$$\ker(2\mathrm{Id} - \mathcal{A}) = \left\{ \underline{X} \in \mathbb{K}^6 : \begin{bmatrix} 0 & & & & & \\ -1 & 0 & & & & \\ 1 & & 0 & & & \\ & -1 & & 0 & & \\ -1 & -1 & -1 & -1 & 0 & \\ & & & & -1 & 3 \end{bmatrix} \underline{X} = 0 \right\}$$

$$= \mathbb{K} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 3 \\ 1 \end{pmatrix} + \mathbb{K} \begin{pmatrix} 0 \\ 0 \\ 1 \\ -1 \\ 0 \\ 0 \end{pmatrix}$$

Consider the equation.

$$(2I_6 - A)^3 (x_1, x_2, 0, \dots)^T = (0, \dots, 0, 3, 1)^T$$

$\Rightarrow x_1 = -3,\ x_2 = 5.$ Then take $v_6 = (-3, 5, 0, 0, 0, 0)^T$

A Jordan basis for $\mathcal{A}$ is

$v_1 = (0, \dots, 1)^T$, $v_2 = (0, 0, 1, -1, 0, 0)^T$, $v_3 = (2\mathrm{Id} - \mathcal{A})^3 v_6$, $v_4 = (2\mathrm{Id} - \mathcal{A})^2 v_6$

$v_5 = (2\mathrm{Id} - \mathcal{A}) v_6$, $v_6 = (-3, 5, 0, 0, 0, 0)^T$

Then $\quad \mathcal{A}(v_1, v_2, \cdots, v_6) = (v_1, v_2, \cdots, v_6)\, J$

$$\Rightarrow \quad J = \begin{bmatrix} & & & & & -3 \\ & & & & 3 & 5 \\ & 1 & & & -3 & \\ & -1 & & -3 & -5 & \\ & & 3 & 5 & -2 & \\ 1 & & 1 & 2 & & \end{bmatrix}^{-1} A \begin{bmatrix} & & & & & -3 \\ & & & & 3 & 5 \\ & 1 & & & -3 & \\ & -1 & & -3 & -5 & \\ & & 3 & 5 & -2 & \\ 1 & & 1 & 2 & & \end{bmatrix}$$

习题 5.2.3. 令 $A = \begin{pmatrix} 0 & 0 & -2 \\ 1 & 0 & 3 \\ 0 & 1 & 0 \end{pmatrix}$. 对每个 $n \in \mathbb{N}$ 求出 A^n.

Let $\mathcal{A}: \mathbb{K}^3 \to \mathbb{K}^3;\ X \mapsto AX$.

$$|\lambda I_3 - A| = \begin{vmatrix} \lambda & & 2 \\ -1 & \lambda & -3 \\ & -1 & \lambda \end{vmatrix} = \lambda(\lambda^2 - 3) + 2 = \lambda^3 - 3\lambda + 2$$
$$= (\lambda - 1)^2 (\lambda + 2)$$

$\Rightarrow I - A = \begin{pmatrix} 1 & & 2 \\ -1 & 1 & -3 \\ & -1 & 1 \end{pmatrix}$ has rank 2.

$\Rightarrow J = \begin{pmatrix} -2 & & \\ & 1 & 1 \\ & & 1 \end{pmatrix}$

$\ker(\mathrm{Id} - \mathcal{A}) = \mathbb{K}(2, -1, -1)^T$

Consider the equation $(I - A)X = (2, -1, -1)^T \Rightarrow \begin{cases} x_1 + 2x_3 = 2 \\ -x_2 + x_3 = -1 \end{cases}$

Then $\xi_3 = (4, 0, -1)^T$ is a solution

② $$-2I_3 - A = \begin{pmatrix} -2 & & 2 \\ -1 & -2 & -3 \\ & -1 & -2 \end{pmatrix}$$

$$\ker(-2Id - A) = \mathbb{K}(1,-2,1)^T$$

③ Let $P = \begin{pmatrix} 1 & 2 & 4 \\ -2 & -1 & 0 \\ 1 & -1 & -1 \end{pmatrix}$.

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 4 & 1 & & \\ -2 & -1 & 0 & & 1 & \\ 1 & -1 & -1 & & & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 2 & 4 & 1 & & \\ & 3 & 8 & 2 & 1 & \\ & -3 & -5 & -1 & & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 2 & 4 & 1 & & \\ & 3 & 8 & 2 & 1 & \\ & & 3 & 1 & 1 & 1 \end{array}\right]$$

$$\to \left[\begin{array}{ccc|ccc} 1 & 2 & & -\frac{1}{3} & -\frac{4}{3} & -\frac{4}{3} \\ & 3 & & -\frac{2}{3} & -\frac{5}{3} & -\frac{8}{3} \\ & & 1 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & & & \frac{1}{9} & -\frac{2}{9} & \frac{4}{9} \\ & 1 & & -\frac{2}{9} & -\frac{5}{9} & -\frac{8}{9} \\ & & 1 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{array}\right]$$

Then $P^{-1} = \begin{bmatrix} \frac{1}{9} & -\frac{2}{9} & \frac{4}{9} \\ -\frac{2}{9} & -\frac{5}{9} & -\frac{8}{9} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{bmatrix}$ and $AP = PJ \Rightarrow A = PJP^{-1}$

$$A^n = PJ^nP^{-1} = \begin{pmatrix} 1 & 2 & 4 \\ -2 & -1 & 0 \\ 1 & -1 & -1 \end{pmatrix} \begin{pmatrix} (-2)^n & & \\ & 1 & n \\ & & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{9} & -\frac{2}{9} & \frac{4}{9} \\ -\frac{2}{9} & -\frac{5}{9} & -\frac{8}{9} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{pmatrix}$$

$$= \frac{1}{9} \begin{pmatrix} 8 - 6n + (-2)^n & 2 - 6n + (-2)^{n+1} & -4 - 6n + (-2)^{n+2} \\ 2 + 3n + (-2)^{n+1} & 5 + 3n + (-2)^{n+2} & 8 + 3n + (-2)^{n+3} \\ -1 + 3n + (-2)^n & 2 + 3n + (-2)^{n+1} & 5 + 3n + (-2)^{n+2} \end{pmatrix}$$

习题 5.2.7. 证明: 任意复方阵 A 与它的转置 A^T 在 $\mathbb{C}$ 上相似.

Pf. Only need to prove for each Jordan block $J_n(\lambda)$, $J_n(\lambda)$ is similar to its transpose.

Since $\operatorname{rank}\left(J_n(0)^T\right)^k = \operatorname{rank} J_n(0)^k$ for any k,

$J_n(\lambda)^T$ has Jordan canonical form $J_n(\lambda)$ as required.

习题 5.2.4. 设 5 阶方阵 A 满足下列条件:

$$\operatorname{rank}(A) = 3\,,\ \operatorname{rank}(A^2) = 2\,,\ \operatorname{rank}(A + I_5) = 4\,,\ \operatorname{rank}(A + I_5)^2 = 3.$$

求 A 的 Jordan 标准形.

$\dim N(A) = 5 - \operatorname{rank}(A) = 2 \to$ 2 Jordan blocks of $\lambda = 0$.

$\dim N(A^2) = 5 - \operatorname{rank}(A^2) = 3 \to$ 1 Jordan block of $\lambda = 0$ and of size ≥ 2

$\dim N(-I_5 - A) = 5 - \operatorname{rank}(A + I_5) = 1 \to$ 1 Jordan block of $\lambda = -1$

$\dim N(-I_5 - A)^2 = 5 - \operatorname{rank}(A + I_5)^2 = 2 \to$ 1 Jordan block of $\lambda = -1$ and of size ≥ 2.

The Jordan canonical form is $\begin{bmatrix} 0 & & & & \\ & 0 & 1 & & \\ & & 0 & & \\ & & & -1 & 1 \\ & & & & -1 \end{bmatrix}$

Advanced Linear Algebra II - HW5

习题 5.2.5. 设 $\mathscr{A}$ 是有限维复向量空间 V 上的线性变换, J 为 $\mathscr{A}$ 的 Jordan 标准形. 设 λ_0 是 $\mathscr{A}$ 的一个特征值, $\mathscr{B} := A - \lambda_0 I$.

1. 对每个 $i \in \mathbb{N}$, 令 $M_i = \ker(\mathscr{B}^i)$, $k := \min\{i \in \mathbb{N} \mid M_i = M_{i+1}\}$. 证明: k 等于 J 中以 λ_0 为特征值的 Jordan 块的最大阶数.
2. 设 k 如前一小题. 令 $N_k = \mathrm{Im}(\mathscr{B}^k)$. 证明 λ_0 不是 $\mathscr{A}|_{N_k}$ 的特征值, 因此 $\mathscr{B}|_{N_k}$ 是可逆变换.
3. 证明 $\dim M_k$ 等于特征值 λ_0 的 (代数) 重数.
4. 设 λ_1 也是 $\mathscr{A}$ 的特征值, $\lambda_1 \neq \lambda_0$. 证明 $G(\lambda_1, \mathscr{A}) \subseteq \mathrm{Im}(\mathscr{B}^k) = N_k$.

1. The maximal size of Jordan blocks in J of e'val λ_0 is the index of nilpotency of $\mathscr{B}$ on $G(\lambda_0, \mathscr{A})$

Since $\mathscr{B}^i|_{G(\lambda_0,\mathscr{A})} = 0 \iff \ker(\mathscr{B}^i) \supseteq G(\lambda_0, \mathscr{A})$, we have

$$\begin{aligned}
\min\{i \mid \mathscr{B}^i|_{G(\lambda_0,\mathscr{A})} = 0\} &= \min\{i \mid \ker(\mathscr{B}^i) \supseteq G(\lambda_0,\mathscr{A})\} \\
&= \min\{i \mid \ker(\mathscr{B}^i) = G(\lambda_0,\mathscr{A})\} \\
&= \min\{i \mid M_i = M_n\} \quad (n = \dim V) \\
&= \min\{i \mid M_i = M_{i+1}\}
\end{aligned}$$

2. Let $v \in N_k = \mathrm{Im}\,\mathscr{B}^k$, then $\exists\, w \in V$ s.t. $v = \mathscr{B}^k w$.

Suppose $\mathscr{A} v = \lambda_0 v$. Then $(\mathscr{A} - \lambda_0) v = \mathscr{B}^{k+1}(w) = 0$

$\Rightarrow w \in \ker \mathscr{B}^{k+1} = \ker \mathscr{B}^k \Rightarrow v = \mathscr{B}^k w = 0$.

$\Rightarrow \lambda_0$ is NOT a e'val of $\mathscr{A}|_{N_0}$.

3. $\dim M_k = \dim G(\lambda_0, \mathscr{A}) =$ alg. mult. of λ_0.

4. Let $v \in G(\lambda_1, \mathscr{A})$ i.e. $(\mathscr{A} - \lambda_1 \mathbb{1})^n v = 0$

Since $\gcd((x-\lambda_0)^k, (x-\lambda_1)^n) = 1$, by Bèzout identity

$\exists\ h(x), g(x)$ s.t. $h(x)(x-\lambda_0)^k + g(x)(x-\lambda_1)^n = 1$.

Thus $(\mathscr{A} - \lambda_0 \mathbb{1})^k h(\mathscr{A}) v + g(\mathscr{A})(\mathscr{A} - \lambda_1 \mathbb{1})^n v = v$, i.e.

$$v = \mathscr{B}^k (h(\mathscr{A}) v) \in \operatorname{Im} \mathscr{B}^k$$

习题 5.2.9. 设 $a_{12}, a_{23}, \cdots, a_{n-1,n}$ 是非零复数. 求复矩阵

Denoted by $A \rightsquigarrow$
$$\begin{pmatrix} a & a_{12} & a_{13} & \cdots & a_{1n} \\ & a & a_{23} & \cdots & a_{2n} \\ & & \ddots & \ddots & \vdots \\ & & & \ddots & a_{n-1,n} \\ & & & & a \end{pmatrix}$$

的 Jordan 标准形.

Method 1.

Let $\mathscr{A}: X \mapsto AX$; $\mathbb{C}^n \to \mathbb{C}^n$ and J be the Jordan canonical form of A

It is obv. that A has only e'val a.

Since $\operatorname{rank}(aI - A) = n-1$, $\dim \ker(a\mathbb{1} - \mathscr{A}) = 1$, that is, J has only one Jordan block.

$\Rightarrow J = J_n(a)$

Method 2:

In $\mathcal{U} = \{\text{upper triangular } M \in \mathrm{Mat}_{n\times n}(\mathbb{C})\}$, we have a decomposition

$$\mathcal{U} = \mathcal{U}_0 \oplus \mathcal{U}_1 \oplus \cdots \oplus \mathcal{U}_{n-1}, \quad \text{where } \mathcal{U}_k = \mathrm{span}\{\mathcal{E}_{i,i+k} : 1 \le i \le n-k\}$$

Moreover, $\mathcal{U}_i \mathcal{U}_j \subseteq \mathcal{U}_{i+j}$ ($\mathcal{U}_i = 0$ for all $i \ge n$),

As a consequence, we have $\mathcal{U}_i^n \subseteq \mathcal{U}_{n\cdot i} = 0$ for all $i \ge 1$.

and similarly

$$\mathcal{U}_{i_1} \mathcal{U}_{i_2} \cdots \mathcal{U}_{i_n} = 0 \quad \text{if} \quad i_j \ne 0 \ \forall j \qquad \cdots\cdots (1)$$

Let $A = A_0 + A_1 + \cdots + A_{n-1}$, where $A_i \in \mathcal{U}_i$. Then

$$A_i A_j \in \mathcal{U}_{i+j}. \qquad \cdots\cdots (2)$$

Observe that $(aI - A)^n = (A_1 + \cdots + A_{n-1})^n = 0$ (by (1)) and

$$(aI - A)^{n-1} = (A_1 + (A_2 + \cdots + A_{n-1}))^{n-1} = A_1^{n-1} \text{ (by (2))}.$$

By a straightforward calculation, we obtain

$$(aI - A)^{n-1} = a_{12} a_{23} \cdots a_{n-1,n} \mathcal{E}_{1,n} \ne 0$$

Thus, the JCF is $\begin{bmatrix} a & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & a \end{bmatrix}$

习题 5.2.10. 设 J 是 n 阶幂零的 Jordan 块. 对每个 $k \in [\![1, n]\!]$ 求出 J^k 的 Jordan 标准形.

Let $J: X \mapsto JX$; $\mathbb{K}^n \to \mathbb{K}^n$

Let $m_k = \lceil \frac{n}{k} \rceil$, in particular $m_1 = n$, $m_n = 1$

向上取整. $\frac{n}{k} \leq m_k < \frac{n}{k} + 1$

Then we have $k m_k \geq n$, $k(m_k - 1) < n \iff k(m_k-1) \leq n-1$.

$$\left.\begin{aligned}(J^k)^{m_k} &= J^{k m_k} = 0 \\ (J^k)^{(m_k-1)} &= J^{k(m_k-1)} \neq 0.\end{aligned}\right\} \Rightarrow X^{m_k} \text{ is a minimal poly of } J^k$$

$$\left.\begin{aligned}\dim \ker J^k &= k \\ \dim \ker J^{k\cdot 2} &= 2k \\ &\vdots \\ \dim \ker J^{k\cdot(m_k-1)} &= k m_k - k \\ \dim \ker J^{k\cdot m_k} &= n.\end{aligned}\right\}$$

$\#\{\text{Jordan blocks of size } i\} = 0$ for all $1 \leq i \leq m_k - 2$.

$\#\{\text{Jordan blocks of size } m_k - 1\}$
$= 2(k m_k - k) - (k m_k - 2k) - n$
$= k m_k - n$

$\#\{\text{Jordan blocks of size } m_k\}$
$= n - k m_k + k$

$\#\{\geq m_k - 2\} = k$

$\#\{\geq m_k - 1\} = k$

$\#\{\geq m_k\} = n - k m_k + k$

Thus. $J = \operatorname{diag}\big(\underbrace{J_{m_k-1}(0), \cdots, J_{m_k-1}(0)}_{k m_k - n}, \underbrace{J_{m_k}(0), \cdots, J_{m_k}(0)}_{n - k m_k + k}\big)$

习题 5.2.13. 求以下矩阵 A 的最小多项式:

$$A=\begin{pmatrix} & & & -a_n \\ 1 & & & -a_{n-1} \\ & \ddots & & \vdots \\ & & 1 & -a_1 \end{pmatrix}.$$

$$|xI-A| = \begin{vmatrix} x & & & a_n \\ -1 & \ddots & & \vdots \\ & \ddots & x & a_2 \\ & & -1 & x+a_1 \end{vmatrix} = \begin{vmatrix} 0 & & & x^n+a_1x^{n-1}+\cdots+a_n \\ -1 & \ddots & & \vdots \\ & \ddots & 0 & x(x+a_1)+a_2 \\ & & -1 & \lambda+a_1 \end{vmatrix}$$

$$= x^n+a_1x^{n-1}+\cdots+a_n$$

Suppose $\lambda\in\mathbb{C}$ is an e'val. Obviously, $\operatorname{rank}(\lambda I-A)=n-1$.

$\Rightarrow$ Geometric mult. of each e'val is 1.

$\Rightarrow$ The minimal poly $f_{\mathbb{C}}(x)$ in $\mathbb{C}$ is just characteristic poly.

Denote by $f_{\mathbb{K}}(x)$ the minimal poly in $\mathbb{K}$. Then $f_{\mathbb{K}}(A)=0$.

By minimality of $f_{\mathbb{C}}(x)$, $f_{\mathbb{C}}(x)\mid f_{\mathbb{K}}(x)$ in $\mathbb{C}$. $\Rightarrow f_{\mathbb{K}}(x)=f_{\mathbb{C}}(x)=x^n+a_1x^{n-1}+\cdots+a_n$

习题 5.2.15. 设 $\mathscr{A}\in\operatorname{End}(V)$ 在 V 的一组有序基 $(\varepsilon_1,\cdots,\varepsilon_n)$ 下的矩阵是 $A=\begin{pmatrix} B & 0 \\ 0 & C\end{pmatrix}$, 其中

$$B=\begin{pmatrix} \lambda_1 & -1 & & \\ & \lambda_1 & \ddots & \\ & & \ddots & -1 \\ & & & \lambda_1 \end{pmatrix}\in\mathbf{M}_r(K),\quad C=\begin{pmatrix} \lambda_2 & 0 & 1 & & \\ & \lambda_2 & 0 & \ddots & \\ & & \ddots & \ddots & 1 \\ & & & \ddots & 0 \\ & & & & \lambda_2 \end{pmatrix}\in\mathbf{M}_{n-r}(K).$$

1. 求 $\mathscr{A}$ 在 V 中的一组 Jordan 基.

2. 求 $\mathscr{A}$ 的最小多项式.

1. 由观察可知, A 的 Jordan 标准形为 $J=\begin{pmatrix} J_r(\lambda_1) & & \\ & J_{\lfloor\frac{n-r}{2}\rfloor}(\lambda_2) & \\ & & J_{\lceil\frac{n-r}{2}\rceil}(\lambda_2)\end{pmatrix}$

助教可以观察👀,
同学们最好是写下过程

故可取基 $\Theta:=\Big(\overbrace{(-1)^{r-1}\varepsilon_1,\cdots,-\varepsilon_{r-1},\varepsilon_r}^{J_r(\lambda_1)},\overbrace{\varepsilon_{r+2},\varepsilon_{r+4},\cdots,\varepsilon_{r+2\lfloor\frac{n-r}{2}\rfloor}}^{J_{\lfloor\frac{n-r}{2}\rfloor}(\lambda_2)},\overbrace{\varepsilon_{r+1},\varepsilon_{r+3},\cdots,\varepsilon_{r-1+2\lceil\frac{n-r}{2}\rceil}}^{J_{\lceil\frac{n-r}{2}\rceil}(\lambda_2)}\Big)$

2.
$$g_{\mathscr{A}}(x)=\begin{cases}(x-\lambda_1)^r(x-\lambda_2)^{\lceil\frac{n-r}{2}\rceil} & 若\ \lambda_1\neq\lambda_2. \\ (x-\lambda_1)^{\max\{r,\lceil\frac{n-r}{2}\rceil\}} & 若\ \lambda_1=\lambda_2.\end{cases}$$

习题 5.2.16. 设 V 是有限维 K-向量空间.

1. 假设线性变换 $\mathscr{D}, \mathscr{D}' \in \mathrm{End}(V)$ 均可对角化, 并且二者的所有特征值 (不计重数意义下) 构成的集合均为 $\{\lambda_1, \cdots, \lambda_r\}$.

   证明: $\mathscr{D} = \mathscr{D}'$ 当且仅当对每个 $i \in [\![1, r]\!]$ 均有 $E(\lambda_i, \mathscr{D}) = E(\lambda_i, \mathscr{D}')$.

2. 假设 $\mathscr{A} \in \mathrm{End}(V)$ 有一种 Jordan–Chevalley 分解式 $\mathscr{A} = \mathscr{D} + \mathscr{N}$, 即, $\mathscr{D} \in \mathrm{End}(V)$ 可对角化, $\mathscr{N} \in \mathrm{End}(V)$ 是幂零变换, 且 $\mathscr{D}, \mathscr{N}$ 可交换. 设 $\lambda_1, \cdots, \lambda_r$ 是 $\mathscr{D}$ 的所有不同特征值.

   (a) 对每个 $i \in [\![1, r]\!]$, 令 $T_i := \mathscr{A} - \lambda_i I$. 证明: $E(\lambda_i, \mathscr{D})$ 是 $\mathscr{N}$ 和 T_i 的不变子空间, $\mathscr{N}|_{E(\lambda_i, \mathscr{D})} = T_i|_{E(\lambda_i, \mathscr{D})}$ 且 $E(\lambda_i, \mathscr{D}) \subseteq G(\lambda_i, \mathscr{A})$.

   (b) 证明: $\lambda_1, \cdots, \lambda_r$ 恰为 $\mathscr{A}$ 的特征多项式的所有不同复数根, 而且 $E(\lambda_i, \mathscr{D}) = G(\lambda_i, \mathscr{A})$.

3. 证明定理 5.2.11 中加法式 Jordan–Chevalley 分解式的唯一性*.

1. D 可对角 $\Rightarrow$ 存在一组基 $\varepsilon = (\varepsilon_1, \cdots, \varepsilon_n)$ 使得 $M_\varepsilon(D)$ 为对角阵.
   即 ε 中均为特征向量. 记 $D\varepsilon_i = \alpha_i \varepsilon_i$, $\alpha_i \in \{\lambda_1, \cdots, \lambda_r\}$

   $D = D' \Leftrightarrow$ 在基 ε 下, $M_\varepsilon(D) = M_\varepsilon(D')$

   $\Leftrightarrow D'\varepsilon_i = D\varepsilon_i = \alpha_i \varepsilon_i \quad \forall i$

   $\Leftrightarrow E(\lambda_i, D') = E(\lambda_i, D)$

2. (a) $\forall v \in E(\lambda_i, D)$, $DNv = NDv = \lambda_i Nv \Rightarrow Nv \in E(\lambda_i, D)$

   $DT_iv = D(N + D - \lambda_i I)v = (N + D - \lambda_i I)Dv = \lambda_i T_i v$

   $\Rightarrow E(\lambda_i, D)$ 为 N 和 T_i 之不变子空间

   $T_i v = (N + D - \lambda_i I)v = Nv + Dv - \lambda_i v = Nv$

   $\Rightarrow T_i|_{E(\lambda_i, D)} = N|_{E(\lambda_i, D)}$

   $(A - \lambda_i I)v = T_i v = Nv$. N 幂零 $\Rightarrow \exists m \in \mathbb{N}$ s.t. $N^m = 0$.

   则 $(A - \lambda_i I)^m v = 0 \Rightarrow v \in G(\lambda_i, A)$

   (b) 若 λ 为 A 一特征值. 即 $\exists w \in V \setminus \{0\}$ s.t. $Aw = (D + N)w = \lambda w$

   若 $N = 0$, 则 λ 为 D 之一特征值 $\Rightarrow \exists i_0$ s.t. $\lambda = \lambda_{i_0}$.

   若 N 幂零且 $\neq 0$, 假设 N 之幂零阶为 k, 则 $N^k w = 0$ 且 $N^{k-1} w \neq 0$.

   则 $N^{k-1} w$ 为 D 之特征向量且特征值为 λ. $\Rightarrow \exists i_0$ s.t. $\lambda = \lambda_{i_0}$.

由于 $\mathcal{N}, \mathcal{D}$ 可交换，$\mathcal{D}$ 与 $\mathcal{A}=\mathcal{D}+\mathcal{N}$ 可交换.

$\Rightarrow G(\lambda_i, \mathcal{A})$ 是 $\mathcal{D}$ 的不变子空间. 故 $\mathcal{D}$ 在 $G(\lambda_i, \mathcal{A})$ 上可对角.

只证. $\mathcal{D}$ 在 $G(\lambda_i, \mathcal{A})$ 上的特征值仅有 λ_i 即可.

若 $\mathcal{D}v=\lambda v$，$v \in G(\lambda_i, \mathcal{A}) \setminus \{0\}$

则 $(\mathcal{A}-\lambda 1)^n v = (\mathcal{D}-\lambda 1+\mathcal{N})^n v = \mathcal{N}^n v = 0$ （$\mathcal{N}$ 幂零）

故 $v \in G(\lambda, \mathcal{A}) \cap G(\lambda_i, \mathcal{A})$ 且 $v \neq 0 \Rightarrow \lambda = \lambda_i$

3. 由于 $\mathcal{A}$ 的广义特征子空间分解唯一（$G(\lambda_i, \mathcal{A}) = \mathrm{Ker}(\mathcal{A}-\lambda_i 1)^n$），

对角部分 $\mathcal{D}$ 唯一. 故 $\mathcal{N} = \mathcal{A} - \mathcal{D}$ 唯一.

习题 5.2.19. 令 $V = K[X]_{\leq n}$. 通过求多项式的形式导数定义线性变换 $\mathscr{D}: V \to V$, 即,

$$\mathscr{D}(1) = 0\,,\quad \text{对 } k \in [\![1, n]\!],\ \ \mathscr{D}(X^k) = kX^{k-1}\,.$$

证明 $\mathscr{D}$ 是一个循环的幂零变换, 并写出它的一组循环基.

X^n, $\mathcal{D}(X^n) = nX^{n-1}$, $\mathcal{D}^2(X) = n(n-1)X^{n-2}$, $\cdots$, $\mathcal{D}^{n-1}(X) = n!\,X$, $\mathcal{D}^n(X) = n!$

上述 $n+1$ 个向量构成 V 的一组基. 故 $\mathcal{D}$ 循环.

且易知 $\mathcal{D}^{n+1} = 0$. 故 $\mathcal{D}$ 幂零.

习题 5.2.20. 设 $\mathscr{A}$ 是 n 维向量空间 V 上的循环幂零变换, $\varepsilon_1, \cdots, \varepsilon_n$ 是它的一组循环基. 试求 $\mathscr{A}$ 的所有不变子空间.

Claim: If W is an invariant subspace of V and $\sum a_i \varepsilon_i \in W$,

then $\varepsilon_k \in W$ for any $\min\{i : a_i \neq 0\} \leq k \leq n$. Denoted by m.

We prove by backward induction on i.

1° $\mathcal{A}^{n-m}\left(\sum_{i=m}^{n} a_i \varepsilon_i\right) = a_m \varepsilon_n \Rightarrow \varepsilon_n \in W$

2° Suppose $\varepsilon_i \in W$ for all $i > k \geq m$.

$\mathcal{A}^{k-m}\left(\sum_{i=m}^{n} a_i \varepsilon_i\right) = a_m \varepsilon_k + a_{m+1}\varepsilon_{k+1} + \cdots \in W \Rightarrow \varepsilon_k \in W$.

Hence, our claim holds.

It is followed that all invariant subspaces are

$W_k = \text{span}\{\varepsilon_i : k \leq i \leq n\}$, for all $1 \leq k \leq n$ and (0).

习题 5.2.21. 设 $\mathscr{A}$ 是有限维向量空间 V 上的幂零变换. 假设 $\mathscr{A}$ 有两个线性无关的特征向量. 证明 $\mathscr{A}$ 不是循环的.

Suppose $\mathscr{A}$ is cyclic with a cyclic basis $v, \mathscr{A}v, \cdots, \mathscr{A}^{n-1}v$.

Let $v_1 = (\sum a_i \mathscr{A}^i)v$, $v_2 = (\sum b_i \mathscr{A}^i)v$ be two eigenvectors with e'val 0 ($\mathscr{A}$ is nilpotent).

$\Rightarrow \mathscr{A}v_1 = \sum_{i=0}^{n-2} a_i \mathscr{A}^{i+1} v = 0 \Rightarrow a_i = 0$ for any $i < n-1$

Similarly, $b_i = 0 \quad \forall i < n-1$

$\Rightarrow v_1 = a_{n-1}\mathscr{A}^{n-1}v$ & $v_2 = b_{n-1}\mathscr{A}^{n-1}v$

v_1 & v_2 are linearly dependent, contrary to our assumption.

习题 5.2.22. 设 V 是 n 维 K-向量空间. 假设线性变换 $\mathscr{A} \in \mathrm{End}(V)$ 的特征多项式 $P_{\mathscr{A}}(X)$ 的所有复数根 $\lambda_1, \cdots, \lambda_r$ 都属于 K. 对于每个 $i \in [\![1, r]\!]$, 记 $m_i = \dim G(\lambda_i, \mathscr{A})$. 假设 $\mathscr{A}$ 的最小多项式等于其特征多项式 $P_{\mathscr{A}}(X)$. 根据命题 5.2.32, 可以取到向量 $v_i \in G(\lambda_i, \mathscr{A})$ 使得 $\mathscr{A}^{m_i-1}v_i, \cdots, \mathscr{A}v_i, v_i$ 构成 $G(\lambda_i, \mathscr{A})$ 的一组基.

1. 对任意多项式 $f \in K[X]$, 证明: 如果 $f(\mathscr{A})v_i = 0$, 则对任何 $u \in G(\lambda_i, \mathscr{A})$ 均有 $f(\mathscr{A})u = 0$.

2. 令 $v = v_1 + \cdots + v_r$.

   证明向量组 $v, \mathscr{A}v, \cdots, \mathscr{A}^{n-1}v$ 是线性无关的. 由此证明 $\mathscr{A}$ 是循环变换.‡.

   (提示: 对任意多项式 $f \in K[X]$, $f(\mathscr{A})v = f(\mathscr{A})v_1 + \cdots + f(\mathscr{A})v_r$, 且每个 $G(\lambda_i, \mathscr{A})$ 都是 $f(\mathscr{A})$ 的不变子空间.)

1. $A^k v_i$, $k \in [\![0, m_i-1]\!]$ 为 $G(\lambda_i, A)$ 的一组基. $f(A)A^k v_i = A^k f(A) v_i = 0$

故 $f(A)=0$.

2. 若 $\sum\limits_{i=0}^{n-1} a_i A^i v = 0$, 即 $\sum a_i A^i v_1 + \sum a_i A^i v_2 + \cdots + \sum a_i A^i v_r = 0$.

由于 $V = \bigoplus\limits_{k=1}^{r} G(\lambda_k, A)$, $\sum\limits_{i=0}^{n-1} a_i A^i v_k = 0 \quad \forall k \in [\![1, r]\!]$

令 $f(A) = \sum\limits_{i=0}^{n-1} a_i A^i$, 则 $f(A)\big|_{G(\lambda_k, A)} = 0$ (由(1))

故 $f(A)=0$. 即 $f(x)$ 为 A 的零化多项式. 而 $\deg f = n-1$.

注意到 A 的最小多项式 $P_A(x)$ 的次数为 n. 故 $f(x)=0$.

即 $\forall i$, $a_i = 0$, 即 $v, Av, \cdots, A^{n-1}v$ 线性无关.

因此 A 循环.

Advanced Linear Algebra II — HW6

[illegible]

习题 6.1.2. 设 $V = \mathbf{M}_n(K)$. 定义

$$\varphi:\ V \times V \longrightarrow K; \quad (A\,,\,B) \longmapsto \mathrm{Tr}(AB)\,.$$

1. 证明 φ 是 V 上的对称双线性型.
2. 令 $n=2$, 取 V 的一组有序基 $\mathcal{B} = (\varepsilon_1,\,\varepsilon_2,\,\varepsilon_3,\,\varepsilon_4)$ 如下:

$$\varepsilon_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad \varepsilon_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \varepsilon_3 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad \varepsilon_4 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

   求 φ 在 $\mathcal{B}$ 下的 Gram 矩阵 $A_1 = \mathcal{M}_{\mathcal{B}}(\varphi)$.
3. 仍设 $n=2$. 另取 V 的一组有序基 $\mathcal{C} = (\eta_1,\,\eta_2,\,\eta_3,\,\eta_4)$ 为

$$\eta_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \eta_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \eta_3 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \eta_4 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

   求 φ 在 $\mathcal{C}$ 下的 Gram 矩阵 $A_2 = \mathcal{M}_{\mathcal{C}}(\varphi)$.
4. 对于前两个小题中的矩阵 A_1 和 A_2, 找出一个可逆矩阵 $P \in \mathbf{M}_4(K)$ 使得 $P^T A_1 P = A_2$.

Pf. 1. ① left-linear: $(a_1A_1 + a_2A_2, B) = \mathrm{Tr}(a_1A_1B + a_2A_2B)$

$$= a_1\mathrm{Tr}(A_1B) + a_2\mathrm{Tr}(A_2B)$$

$$= a_1(A_1,B) + a_2(A_2,B)$$

② Sym: $(B,A) = \mathrm{Tr}(BA) = \mathrm{Tr}(AB) = (A,B)$

$\forall A, A_1, A_2, B \in M_n(\mathbb{K})$ and $a_1, a_2 \in \mathbb{K}$

2. $A_1 = \mathcal{M}_{\mathcal{B}}(\varphi) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

3. $A_2 = \mathcal{M}_{\mathcal{C}}(\varphi) = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & -2 \end{pmatrix}$

4. $\eta_1 = \varepsilon_1 + \varepsilon_4$, $\eta_2 = \varepsilon_1 - \varepsilon_4$, $\eta_3 = \varepsilon_2 + \varepsilon_3$, $\eta_4 = \varepsilon_2 - \varepsilon_3$

即 $(\eta_1, \eta_2, \eta_3, \eta_4) = \mathcal{C} = \mathcal{B} \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 \\ 1 & -1 & 0 & 0 \end{pmatrix}$, 令 $\begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 \\ 1 & -1 & 0 & 0 \end{pmatrix} =: P$

$\forall X, Y \in K^{4\times1}$, $\varphi(\mathcal{C}X, \mathcal{C}Y) = X^T \mathcal{M}_{\mathcal{C}}(\varphi)\, Y = X^T A_2 Y$

同时, $\varphi(\mathcal{C}X, \mathcal{C}Y) = \varphi(\mathcal{B}PX, \mathcal{B}PY) = X^T P^T A P Y$.

由思考题6.3，$A_2 = P^T A_1 P$

习题 6.1.3. 考虑分块对角阵 $A = \begin{pmatrix} A_1 & & & \\ & A_2 & & \\ & & A_3 & \\ & & & A_4 \end{pmatrix}$，其中 A_i 是大小相同的方阵. 令 $A' = \begin{pmatrix} A_4 & & & \\ & A_3 & & \\ & & A_1 & \\ & & & A_2 \end{pmatrix}$. 证明 A 与 A' 相合.

Let A_i be an $n \times n$ matrix.

$$A' = \begin{pmatrix} & & I_n & \\ & I_n & & \\ I_n & & & \\ & & & I_n \end{pmatrix} \begin{pmatrix} A_1 & & & \\ & A_3 & & \\ & & A_4 & \\ & & & A_2 \end{pmatrix} \begin{pmatrix} & & I_n & \\ & I_n & & \\ I_n & & & \\ & & & I_n \end{pmatrix} = \begin{pmatrix} & & I_n & \\ & I_n & & \\ I_n & & & \\ & & & I_n \end{pmatrix} \begin{pmatrix} I_n & & & \\ & & & I_n \\ & & I_n & \\ & I_n & & \end{pmatrix} \begin{pmatrix} A_1 & & & \\ & A_2 & & \\ & & A_4 & \\ & & & A_3 \end{pmatrix} \cdots$$

$$= \begin{pmatrix} & & I_n & \\ & I_n & & \\ I_n & & & \\ & & & I_n \end{pmatrix} \begin{pmatrix} I_n & & & \\ & & & I_n \\ & & I_n & \\ & I_n & & \end{pmatrix} \begin{pmatrix} I_n & & & \\ & I_n & & \\ & & & I_n \\ & & I_n & \end{pmatrix} A \begin{pmatrix} I_n & & & \\ & I_n & & \\ & & & I_n \\ & & I_n & \end{pmatrix} \begin{pmatrix} I_n & & & \\ & & & I_n \\ & & I_n & \\ & I_n & & \end{pmatrix} \begin{pmatrix} & & I_n & \\ & I_n & & \\ I_n & & & \\ & & & I_n \end{pmatrix}$$

习题 6.1.5. (本题有一定的技巧性.)

设 $\langle \cdot, \cdot \rangle$ 是 K-向量空间 V 上的双线性型. 假设对于任意 $u, v \in V$, $\langle u, v \rangle = 0$ 成立时 $\langle v, u \rangle = 0$ 也成立. 证明:

1. 对任意 $x, y, z \in V$ 均有 $\langle x, y \rangle \langle z, x \rangle - \langle x, z \rangle \langle y, x \rangle = 0$.
2. 对任意 $x, y \in V$ 均有 $\langle x, x \rangle \big(\langle x, y \rangle - \langle y, x \rangle \big) = 0$.
3. 假设有向量 $u, v, w \in V$ 满足
$$\langle u, v \rangle \neq \langle v, u \rangle \quad 但 \quad \langle u, w \rangle = \langle w, u \rangle, \ \langle v, w \rangle = \langle w, v \rangle.$$
则
$$\langle u, w \rangle = \langle v, w \rangle = 0, \ \langle u, u \rangle = \langle u + w, u + w \rangle = \langle w, w \rangle = 0.$$
4. 作为双线性型 $\langle \cdot, \cdot \rangle$ 要么是对称的要么交错的.

1. 如果 $\langle x,y\rangle$ 或 $\langle x,z\rangle$ 其中一个为零，等式显然成立.
假设均非零. 即证 $\frac{\langle x,y\rangle}{\langle y,x\rangle} = \frac{\langle x,z\rangle}{\langle z,x\rangle}$，即证 $\langle x, \frac{y}{\langle x,y\rangle} - \frac{z}{\langle z,x\rangle} \rangle = 0$
由于 $\langle \frac{y}{\langle y,x\rangle} - \frac{z}{\langle z,x\rangle}, x \rangle = \frac{\langle y,x\rangle}{\langle y,x\rangle} - \frac{\langle z,x\rangle}{\langle z,x\rangle} = 0$，故上式成立
2. 令1中等式中 $z=x$，即得.
3. 令1中等式 $x=u$，$y=v$，$z=w$，有 $\langle u,v\rangle\langle w,u\rangle - \langle u,w\rangle\langle v,u\rangle = 0$
即 $\langle u,w\rangle(\langle u,v\rangle - \langle v,u\rangle) = 0$. 由题，后一项非零. 故 $\langle u,w\rangle = 0$.
相似地，$\langle v,w\rangle = 0$.

由2，$\langle u,u\rangle(\langle u,v\rangle-\langle v,u\rangle)=0$，又 $\langle u,v\rangle\neq\langle v,u\rangle$，$\langle u,u\rangle=0$.

相似地，$\langle u+w,u+w\rangle(\langle u+w,v\rangle-\langle v,u+w\rangle)=\langle u+w,u+w\rangle(\langle u,v\rangle-\langle v,u\rangle)$

4. 假设 $\langle\cdot,\cdot\rangle$ 不对称，则 $\exists u,v\in V$，s.t. $\langle u,v\rangle\neq\langle v,u\rangle$.

由3知，令 $w=0\in V$，则有 $\langle u,u\rangle=\langle v,v\rangle=0$.

$\forall x\in V$，若 $\langle x,u\rangle\neq\langle u,x\rangle$，则由上述讨论 $\langle x,x\rangle=0$.

若 $\langle x,v\rangle\neq\langle v,x\rangle$，亦有 $\langle x,x\rangle=0$

若 $\langle x,u\rangle=\langle u,x\rangle$ 且 $\langle x,v\rangle=\langle v,x\rangle$，则由3，令 $w=x$，$\langle x,x\rangle=0$.

故 $\forall x\in V$，$\langle x,x\rangle=0$，即 $\langle\cdot,\cdot\rangle$ 交错.

习题 6.1.6. 举例说明: 一个双线性型的左根和右根可以不相等.

令 $V=K^2$，φ 为矩阵 $\begin{pmatrix}1&0\\1&0\end{pmatrix}$ 在标准基下的双线性型.

注意到 $\varphi_{e_1-e_2}(ae_1+be_2)=a-a=0$，$\forall a,b\in K$. 且 $\varphi_{e_1}(e_1)=1\neq0$.

故 $Rad_1(\varphi)=\mathrm{span}\{e_1-e_2\}$.

另一方面 $\varphi'_{e_2}(ae_1+be_2)=0+0=0$，$\forall a,b\in K$. 且 $e_2\notin Rad_1(\varphi)$

故 $Rad_2(\varphi)\neq Rad_1(\varphi)$

习题 6.1.8. 设 $\langle\cdot\,,\,\cdot\rangle$ 是 n 维 K-向量空间 V 上的非退化双线性型, $(v_1,\cdots,v_n)$ 是 V 的一组有序基.

1. 证明: 对任意 $k\in[\![1,n]\!]$ 及任意 $a_1,\cdots,a_k\in K$, 一定存在向量 $w\in V$ 使得
$$\text{对每个 } i\in[\![1,k]\!] \text{ 均有 } \langle v_i\,,\,w\rangle=a_i\ .$$
2. 证明: 若 $k=n$, 则上一小题中的向量 w 是唯一的, 而且只要 $a_1,\cdots,a_n$ 不全为零, 必有 $w\neq0$.

法I. 1. Let G be the Gram matrix of $\langle\,,\,\rangle$ under $(v_1,\cdots,v_n)$.

Let $w=\sum_{i=1}^{n}x_iv_i$. Then $\langle v_i,w\rangle=a_i\ \forall i\in[\![1,k]\!]\iff$

$(I_k\ 0)_{k\times n}G\,X_k=A_k$, where $X_k=(x_1,\cdots,x_k)^T$, $A_k=(a_1,\cdots,a_k)^T$.

Since $\langle , \rangle$ is nondegenerate, $(I_k \ 0)_{k\times n} G$ is full row rank.
Thus, solutions exist. $\Rightarrow$ w exists.

2. If $k=n$, $GX = A$ has a unique solution due to the full rank of G. Thus w is unique. Moreover, $GX=0$ has only the zero solution. The last claim follows.

法II.

1. $\langle \cdot, \cdot \rangle$ 非退化，$\hat{\varphi}': V \to V^{\circ}$ 为双射，考虑 v_i° 之逆象 w_i.
   则 $w = \sum_{i=1}^{k} a_i w_i$ 满足条件.
2. 若有 w' 亦满足，则 $\hat{\varphi}'(w') = \hat{\varphi}'(w) \Rightarrow w' = w$
   由 v_i° 为 V° 之一组基，w_i 为 V 之一组基，故 $w \neq 0$ 如果 a_i 不全为 0.

<u>Remark</u>. This proposition fails in infinite dimensional V.
For instant, $V = \bigoplus_{i=1}^{\infty} \mathbb{C} = \mathrm{span}\{e_i \mid i \in \mathbb{N}^*\}$. (let $e_0 = 0$) with

$$\langle (x_1, \cdots, x_n, \cdots), (y_1, \cdots, y_n, \cdots) \rangle = \sum_{i=1}^{\infty} x_i y_i$$

One can check by definition that $\langle , \rangle$ is a nondeg. sym bil form.

But the basis $(f_i = e_{i-1} + e_i \mid i \in \mathbb{N}^*)$ does NOT have dual basis:

Suppose $h_1 = \sum a_i e_i \in V$ satisfies $\langle f_i, h_1 \rangle = \delta_{i1}$. Then

$a_1 = 1$, $a_1 + a_2 = 0$, $a_2 + a_3 = 0$, $a_3 + a_4 = 0$, $\cdots$

Thus $h_1 = e_1 - e_2 + \cdots + (-1)^{n+1} e_n + \cdots \notin V$. contrary to our assumption.

习题 6.1.9. 设 φ 是 K-向量空间 V 上的对称或交错双线性型, U, W 是 V 的子空间.

1. 证明: 如果 $V = U + W$ 并且 U 和 W 关于 φ 正交, 那么 $\mathrm{Rad}(\varphi) = \mathrm{Rad}(\varphi|_U) + \mathrm{Rad}(\varphi|_W)$.
2. 假设 $V = U + U^{\perp}$. 证明: φ 是非退化的当且仅当 $\varphi|_U$ 和 $\varphi|_{U^{\perp}}$ 都是非退化的.

证: 1. $\forall u \in \mathrm{Rad}(\varphi|_u)$, $w \in \mathrm{Rad}(\varphi|_w)$, $v \in V$. $\exists v_1 \in U$, $v_2 \in W$ s.t. $v = v_1 + v_2$

$\varphi(u+w, v) = \varphi(u, v_1+v_2) + \varphi(w, v_1+v_2) = 0$

故 $\mathrm{Rad}(\varphi|_u) + \mathrm{Rad}(\varphi|_w) \subseteq \mathrm{Rad}(\varphi)$

反过来. $\forall \hat{v} \in \mathrm{Rad}\,\varphi \subseteq V$, $\exists \hat{u} \in U$, $\hat{w} \in W$ s.t. $\hat{v} = \hat{u} + \hat{w}$.

$\forall u \in U$, $w \in W$, $\varphi(\hat{u}, u) = \varphi(\hat{u}+\hat{w}, u) = 0 \Rightarrow \hat{u} \in \mathrm{Rad}(\varphi|_u)$

$\varphi(\hat{w}, w) = \varphi(\hat{w}+\hat{u}, w) = 0 \Rightarrow \hat{w} \in \mathrm{Rad}(\varphi|_w)$

故 $\mathrm{Rad}(\varphi) \supseteq \mathrm{Rad}(\varphi|_u) + \mathrm{Rad}(\varphi|_w)$

证毕.

2. 由 1, $\mathrm{Rad}(\varphi) = \phi$ iff $\mathrm{Rad}(\varphi|_u)$ & $\mathrm{Rad}(\varphi|_w) = \phi$.

习题 6.1.10. 设 φ 是由如下表达式给出的 K^4 上的双线性型:

$$\varphi(u, v) = -x_1y_1 + 2x_1y_2 + 2x_2y_1 - 3x_2y_2 + x_2y_4 - 2x_1y_4 + x_4y_2 - 2x_4y_1 + 2x_4y_4,$$

其中 $u = (x_1, x_2, x_3, x_4)^T$, $v = (y_1, y_2, y_3, y_4)^T$. 令 $W = \mathrm{span}(e_1, e_2) \subseteq K^4$.

1. 求 $W^{\perp}$.
2. 证明 $\varphi|_W$ 是非退化的.
3. 将 $v = (1, 2, 3, 4)^T$ 分解为 $v = w + w'$ 的形式, 其中 $w \in W$, $w' \in W^{\perp}$.

易知 φ 对称.

1. 设 $v = (y_1, y_2, y_3, y_4)^T \in W^{\perp}$, 则 $\varphi(e_1, v) = -y_1 + 2y_2 - 2y_4 = 0$ & $\varphi(e_2, v) = 2y_1 - 3y_2 + y_4 = 0$

$\Rightarrow W^{\perp} = \mathrm{span}\{4e_1 + 3e_2 + e_4, e_3\}$

2. $\varphi|_W(x, v) = -x_1y_1 + 2x_1y_2 + 2x_2y_1 - 3x_2y_2 = (x_1, x_2)\begin{pmatrix} -1 & 2 \\ 2 & -3 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$

$\begin{pmatrix} -1 & 2 \\ 2 & -3 \end{pmatrix}$ 满秩.

3. $v = 3e_3 + 4(4e_1 + 3e_2 + e_4) - 15e_1 - 10e_2$

习题 6.1.11. 设 φ 为 K-向量空间 V 上的双线性型, M 是 V 的子空间. 定义

$$\mathcal{L}(M) = \{v \in V \mid \forall\, x \in M,\ \varphi(v, x) = 0\}\,; \quad \mathcal{R}(M) = \{v \in V \mid \forall\, x \in M,\ \varphi(x, v) = 0\}\,.$$

1. 证明 $\mathcal{L}(M)$ 和 $\mathcal{R}(M)$ 都是 V 的子空间.
2. 设 $N \subseteq V$ 也是一个子空间. 证明:
$$\mathcal{L}(M+N) = \mathcal{L}(M) \cap \mathcal{L}(N), \quad \mathcal{R}(M+N) = \mathcal{R}(M) \cap \mathcal{R}(N).$$
3. 假设 V 是有限维的, 并且 φ 是非退化的. 证明:
$$\dim \mathcal{L}(M) = \dim \mathcal{R}(M) = \dim V - \dim M, \quad \mathcal{R}(\mathcal{L}(M)) = \mathcal{L}(\mathcal{R}(M)) = M.$$
4. 假设 V 是有限维的, 并且 φ 是非退化的. 证明: 对任意子空间 $M, N \subseteq V$,
$$\mathcal{L}(M \cap N) = \mathcal{L}(M) + \mathcal{L}(N), \quad \mathcal{R}(M \cap N) = \mathcal{R}(M) + \mathcal{R}(N)\,.$$

$L(R(M)+R(N))$

$= \mathcal{L}(R(M)) \cap \mathcal{L}(R(N))$

$= M \cap N$

$= L(R(M \cap N))$

Pf. 1. ① $0 \in \mathcal{L}(M)$

② $\forall v, w \in \mathcal{L}(M)$, $\varphi(v+w, x) = \varphi(v,x) + \varphi(w,x) = 0$, $\forall x \in M$.

$\Rightarrow v+w \in \mathcal{L}(M)$

③ $\forall k \in \mathbb{K}$, $v \in \mathcal{L}(M)$. $\varphi(kv, x) = k\varphi(v,x) = 0$, $\forall x \in M \Rightarrow kv \in \mathcal{L}(M)$.

In conclusion, $\mathcal{L}(M)$ is a subspace, similarly, so is $R(M)$

2. $\mathcal{L}(M+N) \subseteq \mathcal{L}(M) \cap \mathcal{L}(N)$:

$\forall v \in \mathcal{L}(M+N)$, $\varphi(v, m) = 0$ and $\varphi(v, n) = 0$ $\forall m \in M, n \in N$

$\mathcal{L}(M+N) \supseteq \mathcal{L}(M) \cap \mathcal{L}(N)$.

$\forall v \in \mathcal{L}(M) \cap \mathcal{L}(N)$, $\forall x \in M+N$, $\exists m \in M$ & $n \in N$ s.t. $x = m+n$.

and $\varphi(v, x) = \varphi(v, m) + \varphi(v, n) = 0$.

In conclusion, $\mathcal{L}(M+N) = \mathcal{L}(M) + \mathcal{L}(N)$. Similarly, we have $R(M+N) = R(M) + R(N)$.

3. ① Consider $\varphi_1 : V \to M^\circ$ by $\varphi_1(v) = \varphi(v, \cdot)|_M$.

View $M^\circ = \mathrm{Hom}(M, \mathbb{K})$ as a subspace of V° by extension by 0.

Then φ_1 is surjective since $\varphi : V \to V^\circ$ is bijection.

Thus $\ker \varphi_1 = \mathcal{L}(M)$, $\mathrm{im}\,\varphi_1 = M^\circ$.

By rank-nullity theorem,

$$\dim V = \dim M^\circ + \dim \mathcal{L}(M) = \dim M + \dim \mathcal{L}(M)$$

Similarly, $\dim V = \dim M + \dim R(M)$. $\Rightarrow \dim R(M) = \dim \mathcal{L}(M)$

③. $\forall x \in M$, $\forall w \in \mathcal{L}(M)$, $\varphi(w,x) = 0 \Rightarrow x \in R(\mathcal{L}(M))$

Thus $M \subseteq R(\mathcal{L}(M))$.

On the other hand, $\dim R(\mathcal{L}(M)) = \dim V - \dim \mathcal{L}(M) = \dim M$.

$\Rightarrow M = R(\mathcal{L}(M))$.

Similarly, we have $M = \mathcal{L}(R(M))$

4. Thanks to 2 & 3,

$$R(\mathcal{L}(M) + \mathcal{L}(N)) = R\mathcal{L}(M) \cap R\mathcal{L}(N) = M \cap N$$

$$\Rightarrow \mathcal{L}(M) + \mathcal{L}(N) = \mathcal{L}R(\mathcal{L}(M) + \mathcal{L}(N)) = \mathcal{L}(M \cap N)$$

Similarly, we have $R(M) + R(N) = R(M \cap N)$

习题 6.1.12. 设 $f : V \to K$ 和 $g : V \to K$ 是 K-向量空间 V 上的线性函数. 假设对于任意 $v \in V$ 均有 $f(v)g(v) = 0$.

证明 $f = 0$ 或 $g = 0$.

Pf. $\{v \in V : f(v)g(v) = 0\} = \ker f \sqcup \ker g = V$

Since $\ker f$ and $\ker g$ are subspaces of V, $\ker f = V$ or $\ker g = V$, i.e. $f = 0$ or $g = 0$.

<u>Remark</u>. It is also true for finite many f and g.

习题 6.1.13. 设 φ 是 K-向量空间 V 上的对称双线性型. 假设存在 V 上的线性函数 $f, g : V \to K$ 使得

$$\text{对任意 } u, v \in V, \ \varphi(u, v) = f(u)g(v).$$

证明: 存在线性函数 $\ell : V \to K$ 以及非零常数 $\lambda \in K$ 使得

$$\text{对任意 } u, v \in V, \ \varphi(u, v) = \lambda \ell(u)\ell(v).$$

(提示: 若 $f \neq 0$, $g \neq 0$, 可以证明存在非零常数 $\lambda \in K$ 使得 $g = \lambda f$.)

Since φ is sym, $\varphi(u, \cdot) = \varphi(\cdot, u)$, $\forall u \in V$, i.e.

$$f(u) g = g(u) f. \quad \forall u \in V.$$

Suppose $f \neq 0$ & $g \neq 0$ (otherwise, it's trivial). Then $\exists u \in V$ s.t. $f(u) \neq 0$.

Thus $g = \frac{g(u)}{f(u)} f$. Note that $g(u) \neq 0$ since $g = 0 \cdot f = 0$ otherwise.

Then $\varphi = \frac{g(u)}{f(u)} f \cdot f$ as desired.

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习题 6.2.2. 对于下列二次型 $q \in \mathrm{Quad}(K^4)$, 求出 q 在 K^4 的标准基下的 Gram 矩阵以及极化型 b_q 的表达式:

(1) $q = -x_1x_3 - 2x_1x_4 + x_3^2 - 5x_3x_4$;

(2) $q = -x_2^2 - x_3^2 - x_1x_4$;

(3) $q = x_1x_2 + x_1x_3 + x_1x_4 + x_2x_3 + x_2x_4 + x_3x_4$.

1. $\frac{1}{2}(q(u+v) - q(u) - q(v)) = \frac{1}{2}\Big[-(x_1+y_1)(x_3+y_3) - 2(x_1+y_1)(x_4+y_4) + (x_3+y_3)^2$
$- 5(x_3+y_3)(x_4+y_4) + x_1x_3 + 2x_1x_4 - x_3^2 + 5x_3x_4 + y_1y_3 + 2y_1y_4 - y_3^2 + 5y_3y_4\Big]$

$$= \frac{1}{2}\left(-x_1y_3 - x_3y_1 - 2x_1y_4 - 2x_4y_1 + 2x_3y_3 - 5x_3y_4 - 5x_4y_3\right)$$

$$b_q(u,v) = \frac{1}{2}\left(-x_1y_3 - x_3y_1 - 2x_1y_4 - 2x_4y_1 + 2x_3y_3 - 5x_3y_4 - 5x_4y_3\right)$$

Gram matrix is $A = \begin{bmatrix} 0 & 0 & -\frac{1}{2} & -1 \\ 0 & 0 & 0 & 0 \\ -\frac{1}{2} & 0 & 1 & -\frac{5}{2} \\ -1 & 0 & -\frac{5}{2} & 0 \end{bmatrix}$

2. $\frac{1}{2}(q(u+v) - q(u) - q(v)) = \frac{1}{2}\left[(x_2+y_2)^2 - (x_3+y_3)^2 - (x_1+y_1)(x_4+y_4) + x_2^2 + x_3^2 + x_1x_4 + y_2^2 + y_3^2 + y_1y_4\right]$
$= -x_2y_2 - x_3y_3 - \frac{1}{2}x_1y_4 - \frac{1}{2}y_1x_4$

$b_q : K^4 \times K^4 \to K$; $(u,v) \longmapsto -x_2y_2 - x_3y_3 - \frac{1}{2}x_1y_4 - \frac{1}{2}y_1x_4$

Gram 矩阵: $A = \begin{pmatrix} 0 & & & -\frac{1}{2} \\ & -1 & & \\ & & -1 & \\ -\frac{1}{2} & & & 0 \end{pmatrix}$

3. $\frac{1}{2}(q(u+v) - q(u) - q(v)) = \Big[(x_1+y_1)(x_2+y_2) + (x_1+y_1)(x_3+y_3) + (x_1+y_1)(x_4+y_4) + (x_2+y_2)(x_3+y_3)$
$+ (x_2+y_2)(x_4+y_4) + (x_3+y_3)(x_4+y_4) - x_1x_2 - x_1x_3 - x_1x_4 - x_2x_4 - x_3x_4$
$- x_2x_3 - y_1y_2 - y_1y_3 - y_1y_4 - y_2y_3 - y_2y_4 - y_3y_4\Big] \cdot \frac{1}{2}$

$= \frac{1}{2}\big(x_1y_2 + y_2x_1 + x_1y_3 + x_3y_1 + x_1y_4 + x_4y_1 + x_2y_3 + x_3y_2 + x_2y_4 + x_4y_2$
$+ x_3y_4 + x_4y_3\big)$

$b_q : K^4 \times K^4 \to K$; $(u,v) \mapsto \frac{1}{2}\sum_{i \neq j} x_iy_j$, Gram 矩阵: $A = \frac{1}{2} \cdot \begin{pmatrix} & 1 & 1 & 1 \\ 1 & & 1 & 1 \\ 1 & 1 & & 1 \\ 1 & 1 & 1 & \end{pmatrix}$

习题 6.2.3. 找出适当的可逆线性变量替换将下列二次型化为标准形:

(1) $q = -4x_1x_2 + 2x_1x_3 + 2x_2x_3$;

(2) $q = x_1^2 + 2x_1x_2 + 2x_2^2 + 4x_2x_3 + 4x_3^2$;

(3) $q = x_1^2 - 3x_2^2 - 2x_1x_2 + 2x_1x_3 - 6x_2x_3$;

(4) $q = 8x_1x_4 + 2x_3x_4 + 2x_2x_3 + 8x_2x_4$.

3)
$$\begin{cases} y_1 = x_1 - x_2 + x_3 \\ y_2 = 2x_2 + x_3 \\ y_3 = x_3 \end{cases}$$

$$q = (x_1 - x_2 + x_3)^2 - 4x_2^2 - x_3^2 - 4x_2x_3$$
$$= (x_1 - x_2 + x_3)^2 - (2x_2 + x_3)^2$$
$$= y_1^2 - y_2^2$$

(4)
$$\begin{cases} x_1 = x_1' - x_4' \\ x_2 = x_2' \\ x_3 = x_3' \\ x_4 = x_1' + x_4' \end{cases}$$

$$\Rightarrow q = 8x_1'^2 - 8x_4'^2 + 2x_1'x_3' + 2x_3'x_4' + 2x_2'x_3' + 8x_1'x_2' + 8x_2'x_4'$$
$$= 8\left(x_1' + \frac{x_2'}{2} + \frac{x_3'}{8}\right)^2 - 2x_2'^2 - \frac{x_3'^2}{8} + x_2'x_3' + 2x_3'x_4' + 8x_2'x_4' - 8x_4'^2$$
$$= 8\left(x_1' + \frac{x_2'}{2} + \frac{x_3'}{8}\right)^2 - 2\left(x_2' - \frac{1}{4}x_3' - 2x_4'\right)^2 + 4x_3'x_4'$$
$$= 8\left(x_1' + \frac{x_2'}{2} + \frac{x_3'}{8}\right)^2 - 2\left(x_2' - \frac{1}{4}x_3' - 2x_4'\right)^2 + (x_3' + x_4)^2 - (x_3' - x_4')^2$$

令
$$\begin{cases} y_1 = x_1' + \frac{x_2'}{2} + \frac{x_3'}{8} \\ y_2 = x_2' - \frac{1}{4}x_3' - 2x_4' \\ y_3 = x_3' + x_4' \\ y_4 = x_3' - x_4' \end{cases}$$
, 则 $q = 8y_1^2 - 2y_2^2 + y_3^2 - y_4^2$

习题 6.2.7. 设 $A=(a_{ij})$ 为 $s\times n$ 实矩阵. 考虑实二次型

$$q=\sum_{i=1}^{s}(a_{i1}x_1+a_{i2}x_2+\cdots+a_{in}x_n)^2\,.$$

证明 $\operatorname{rank}(q)=\operatorname{rank}(A)$.

证: $q = X^TA^TAX$. 故等价于证 $\operatorname{rank} A^TA = \operatorname{rank} A$.

只证: $N(A^TA)=N(A)$ (高代I: $N(M)=Sol(M;0)$)

"$\supseteq$": 显然. "$\subseteq$": $\forall X\in\mathbb{R}^{n\times1}$ s.t. $A^TAX=0 \Rightarrow X^TA^TAX=|AX|=0$

由 $AX\in\mathbb{R}^{n\times1}$, $AX=0$.

习题 6.2.11. 设 q 是 n 维实向量空间 V 上的二次型. 定义 $C(q):=\{v\in V\,|\,q(v)=0\}$.

1. 证明: $C(q)$ 是 V 的子空间当且仅当 q 是半正定或半负定的.
2. 假设 $C(q)$ 是个子空间. 求它的维数.
3. 对于一般的情况, 设 q 的秩为 r, 正、负惯性指数分别为 p, s.
   证明 $C(q)$ 中能够包含的子空间维数最大值是 $n-\max\{p,s\}=\min\{p,s\}+n-r$.

证: 1. "$\Rightarrow$" Suppose q is NOT definite, i.e. under a basis $(v_1,\cdots,v_n)$,

Gram matrix is $\begin{pmatrix} I_p & & \\ & -I_q & \\ & & 0_r\end{pmatrix}$ with $p,q\geq 1$.

Then $q(v_1-v_{p+1})=q(v_1+v_{p+1})=1$.

Since $C(q)$ is a subspace, $v_1, v_{p+1}\in C(q)$, contradiction!

"$\Leftarrow$" 令 q 在一组基 $v_1,\cdots,v_n$ 下的 Gram 矩阵为 $A=\begin{pmatrix} I_r & \\ & 0\end{pmatrix}$ (或 $\begin{pmatrix} -I_r & \\ & 0\end{pmatrix}$)

则 $\forall X\in\mathbb{R}^{n\times1}$, $\exists X_1\in\mathbb{R}^{r\times1}$, $X_2\in\mathbb{R}^{(n-r)\times1}$ s.t. $X=\begin{pmatrix}X_1\\X_2\end{pmatrix}$.

$X^TAX=\begin{pmatrix} X_1^TX_1 & \\ & 0\end{pmatrix}=0$ (或 $\begin{pmatrix} -X_1^TX_1 & \\ & 0\end{pmatrix}=0$) $\Longleftrightarrow X_1=0$

故 $C(q)=\left\{(v_1,\cdots,v_n)\begin{pmatrix}0\\X_2\end{pmatrix}: X_2\in\mathbb{R}^{(n-r)\times1}\right\}$ 是一个子空间.

2. 由1, $\dim C(q)=n-r$

3. 假设 U 为 $C(q)$ 中维数 $>n-\max\{p,s\}$ 的子空间.

令 W 为一个维数为 $\max\{p,s\}$ 的正定或负定的子空间, (一定存在, 取标准型下的一组基的子向量组即可)

由于 $\dim U+\dim W>n$, 则 $U\cap W\neq\emptyset$ 矛盾!

习题 6.2.19. 设 $A \in \mathbf{M}_n(\mathbb{R})$ 是对称矩阵. 证明: 对于任何充分大的实数 t, 矩阵 $tI_n + A$ 是正定矩阵.

Pf. When $n=1$, it is obv. It allows us to use induction on n.

Suppose $t_1 I_{n-1} + A_{n-1}$ is positive definite. where A_{n-1} is $(n-1)$-principle submatrix. Thus all principle minors of $t_1 I - A$ except $|t_1 I - A|$ are >0.

Consider $|tI-A|$ which is a polynomial on t with leading term t^n. Thus $\exists$ sufficiently large $t_2 > 0$, s.t. $|t_2 I - A| > 0$.

Then taking $t = \max\{t_1, t_2\}$, all principle minors of $tI-A$ are >0 $\iff$ $tI-A$ is positive definite

习题 6.2.23. 假设 $A = (a_{ij})$ 为 n 阶正定实矩阵. 定义

$$q : \mathbb{R}^n \longrightarrow \mathbb{R}\,; \quad y = (y_1, \cdots, y_n)^T \longmapsto \det \begin{pmatrix} A & y \\ y^T & 0 \end{pmatrix}.$$

1. 证明: q 是 $\mathbb{R}^n$ 上的二次型, 并且 q 是负定的.

2. 设 P_{n-1} 是 A 的 $n-1$ 阶顺序主子式. 证明: $|A| \le a_{nn} P_{n-1}$.

3. 证明: $|A| \le a_{11} a_{22} \cdots a_{nn}$.

4. 证明: 对于任意可逆矩阵 $P = (p_{ij}) \in \mathbf{M}_n(\mathbb{R})$ 均有

$$|P|^2 \le \prod_{i=1}^{n} (p_{1i}^2 + p_{2i}^2 + \cdots + p_{ni}^2)\,.$$

Pf 1. $q(cy) = \det\begin{pmatrix} A & cy \\ cy^T & 0 \end{pmatrix} = c^2 \det\begin{pmatrix} A & y \\ y^T & 0 \end{pmatrix} = c^2 q(y)$

$q(e_i) = (-1)^{n+i+1}(-1)^{n+i} A\begin{Bmatrix} 1 \cdots \hat{i} \cdots n \\ 1 \cdots \hat{i} \cdots n \end{Bmatrix} = -A\begin{Bmatrix} 1 \cdots \hat{i} \cdots n \\ 1 \cdots \hat{i} \cdots n \end{Bmatrix}$

$b_q(x,y) = \frac{1}{2}\left(q(x+y) - q(x) - q(y)\right) = \frac{1}{2}\left(\det\begin{pmatrix} A & x+y \\ x^T+y^T & 0 \end{pmatrix} - \det\begin{pmatrix} A & x \\ x^T & \end{pmatrix} - \det\begin{pmatrix} A & y \\ y^T & 0 \end{pmatrix}\right)$

$$= \frac{1}{2}\left(\det\begin{pmatrix} A & X \\ Y^T & 0 \end{pmatrix} + \det\begin{pmatrix} A & Y \\ X^T & \end{pmatrix}\right) = \det\begin{pmatrix} A & X \\ Y^T & \end{pmatrix}$$

$$b_q(e_i, e_j) = (-1)^{n+1+i}(-1)^{n+j} A\begin{Bmatrix} 1 \cdots \hat{i} \cdots n \\ 1 \cdots \hat{j} \cdots n \end{Bmatrix}$$

$$= -A\begin{Bmatrix} 1 \cdots \hat{i} \cdots n \\ 1 \cdots \hat{j} \cdots n \end{Bmatrix}$$

$$= -\|A\| (A^{-1})_{ji}$$

$\Rightarrow$ Gram matrix $= -\|A\| (A^{-1})^T = -\|A\| A^{-1}$

Suppose $A = P^T P$, then $A^{-1} = P^{-1} \cdot P^{-1\,T}$ is still positive definite

$\Rightarrow$ Gram of b_q is negative definite.

2.

$$A = \begin{pmatrix} A_{n-1} & \alpha \\ \alpha^T & a_{nn} \end{pmatrix} = \begin{pmatrix} A_{n-1} & \\ & 1 \end{pmatrix}\begin{pmatrix} I_{n-1} & A_{n-1}^{-1}\alpha \\ \alpha^T & a_{nn} \end{pmatrix}$$

$$= \begin{pmatrix} A_{n-1} & \\ & 1 \end{pmatrix}\begin{pmatrix} I_{n-1} & \\ \alpha^T & 1 \end{pmatrix}\begin{pmatrix} I_{n-1} & A_{n-1}^{-1}\alpha \\ & a_{nn} - \alpha^T A_{n-1}^{-1}\alpha \end{pmatrix}$$

$$\Rightarrow |A| = |A_{n-1}| \cdot (a_{nn} - \alpha^T A_{n-1}^{-1}\alpha)$$

$$= a_{nn} P_{n-1} - P_{n-1}(\alpha^T A_{n-1}^{-1}\alpha)$$

$$\leq a_{nn} P_{n-1}$$ since A_{n-1}^{-1} is positive definite.

3. Prove by iterating 2).

4. $LHS = (P^TP)_{11}(P^TP)_{22} \cdots (P^TP)_{nn} \geq |P^TP| = |P|^2$

习题 7.1.4. 设 u, v 是内积空间 V 中的向量. 证明下列条件等价:

1. u 与 v 正交.
2. 对所有 $a \in \mathbb{R}$ 均有 $\|u\| \le \|u + av\|$.

"$\Rightarrow$" $\langle u+av, u+av\rangle = \langle u,u\rangle + 2a\langle u,v\rangle + a^2\langle v,v\rangle$

$= \langle u,u\rangle + a^2\langle v,v\rangle \geqslant \langle u,u\rangle$

"$\Leftarrow$" $\|u\| \leqslant \|u+av\| \Leftrightarrow a^2\langle v,v\rangle + 2a\langle u,v\rangle = a(a\langle v,v\rangle + 2\langle u,v\rangle) \geqslant 0 \quad \forall a \in \mathbb{R}$

$\Rightarrow \langle u,v\rangle = 0$ (二次函数)

习题 7.1.8. 设 $V = \mathbf{M}_n(\mathbb{R})$. 定义 V 上的双线性型

$$\langle \cdot, \cdot \rangle : V \times V \longrightarrow \mathbb{R}; \quad (A, B) \longmapsto \langle A, B \rangle := \mathrm{Tr}(A^T B).$$

证明这个双线性型是 V 上的一个内积. 该内积称为 $\mathbf{M}_n(\mathbb{R})$ 上的 Frobenius[†]**内积**.

证: 1. 对称: $\langle B, A\rangle = \mathrm{Tr}(B^TA) = \mathrm{Tr}((B^TA)^T) = \mathrm{Tr}(A^TB) = \langle A, B\rangle$

2. 正定: $\langle A, A\rangle = \mathrm{Tr}(A^TA) = \sum_{ij} a_{ij}^2 \geqslant 0$ 其中 $A = (a_{ij})$

且 当且仅当 $A=0$ 取"="

习题 7.1.12. 假设 $\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4, \varepsilon_5$ 是五维内积空间 V 中的一组规范正交基. 令

$$\alpha_1 = \varepsilon_1 + \varepsilon_3\,,\ \alpha_2 = \varepsilon_1 - \varepsilon_2 + \varepsilon_4\,,\ \alpha_3 = 2\varepsilon_1 + \varepsilon_2 + \varepsilon_3\,.$$

求子空间 $U := \mathrm{span}(\alpha_1, \alpha_2, \alpha_3)$ 的一组规范正交基.

证: $\langle \varepsilon_1+\varepsilon_3, \varepsilon_1+\varepsilon_3\rangle = 2 \Rightarrow$ 令 $\beta_1 = \frac{\alpha_1}{\sqrt{2}}$

$\langle \alpha_2, \beta_1\rangle = \frac{1}{\sqrt{2}}$. $\Rightarrow$ 令 $\alpha_2' = \alpha_2 - \frac{1}{\sqrt{2}}\beta_1$. $\langle \alpha_2', \alpha_2'\rangle = \frac{5}{2}$

$\Rightarrow$ 令 $\beta_2 = \frac{\alpha_2'}{\sqrt{\langle \alpha_2', \alpha_2'\rangle}} = \frac{\sqrt{10}}{5}\alpha_2' = \frac{\sqrt{10}}{5}\alpha_2 - \frac{\sqrt{5}}{5}\beta_1 = \frac{\sqrt{10}}{5}\alpha_2 - \frac{\sqrt{10}}{10}\alpha_1$

$\langle \alpha_3, \beta_1\rangle = \frac{1}{\sqrt{2}}\langle \alpha_3, \alpha_1\rangle = \frac{3}{\sqrt{2}}$

$\langle \alpha_3, \beta_2\rangle = \frac{\sqrt{10}}{5}\langle \alpha_3, \alpha_2\rangle - \frac{\sqrt{10}}{10}\langle \alpha_3, \alpha_1\rangle = \frac{\sqrt{10}}{5}\cdot 1 - \frac{\sqrt{10}}{10}\cdot 3 = -\frac{\sqrt{10}}{10}$

$\alpha_3' = \alpha_3 - \langle \alpha_3, \beta_1\rangle\beta_1 - \langle \alpha_3, \beta_2\rangle\beta_2$

$= \alpha_3 - \frac{3}{\sqrt{2}}\beta_1 + \frac{\sqrt{10}}{10}\beta_2$

$\langle \alpha_3', \alpha_3'\rangle = \langle \alpha_3, \alpha_3\rangle + \frac{9}{2} + \frac{1}{10} - 3\sqrt{2}\langle \alpha_3, \beta_1\rangle + \frac{\sqrt{10}}{5}\langle \alpha_3, \beta_2\rangle$

$= 6 + \frac{9}{2} + \frac{1}{10} - 9 - \frac{1}{5} = \frac{7}{5}$

令 $\beta_3 = \alpha_3' / \sqrt{\langle \alpha_3', \alpha_3'\rangle} = \frac{\sqrt{35}}{7}\alpha_3 - \frac{3\sqrt{70}}{14}\beta_1 + \frac{1}{\sqrt{14}}\beta_2$

$$= \frac{\sqrt{35}}{7}\alpha_3 - \frac{3\sqrt{35}}{14}\alpha_1 + \frac{1}{\sqrt{14}}\cdot\frac{\sqrt{10}}{5}\alpha_2 - \frac{1}{\sqrt{14}}\frac{\sqrt{10}}{10}\alpha_1$$

$$= -\frac{8}{\sqrt{35}}\alpha_1 + \frac{1}{\sqrt{35}}\alpha_2 + \frac{\sqrt{35}}{7}\alpha_3$$

$$(\beta_1, \beta_2, \beta_3) = (\alpha_1, \alpha_2, \alpha_3)\begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{10}} & -\frac{8}{\sqrt{35}} \\ 0 & \frac{\sqrt{10}}{5} & \frac{1}{\sqrt{35}} \\ 0 & 0 & \frac{\sqrt{35}}{7} \end{pmatrix}$$

$$= (\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4, \varepsilon_5)\begin{pmatrix} 1 & 1 & 2 \\ 0 & -1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{\sqrt{10}}{10} & -\frac{8}{\sqrt{35}} \\ & \frac{\sqrt{10}}{5} & \frac{1}{\sqrt{35}} \\ & & \frac{\sqrt{35}}{7} \end{pmatrix}$$

$$= (\varepsilon_1, \cdots, \varepsilon_5)\begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{\sqrt{10}}{10} & \frac{3\sqrt{35}}{35} \\ 0 & -\frac{\sqrt{10}}{5} & \frac{4\sqrt{35}}{35} \\ \frac{1}{\sqrt{2}} & -\frac{\sqrt{10}}{10} & -\frac{3\sqrt{35}}{35} \\ 0 & \frac{\sqrt{10}}{5} & \frac{1}{\sqrt{35}} \\ 0 & 0 & 0 \end{bmatrix}$$

习题 7.1.16. 设 $\alpha_1, \cdots, \alpha_r$ 是内积空间 V 中的一组向量. 令 $A = (a_{ij}) \in \mathbf{M}_r(\mathbb{R})$, 其中 $a_{ij} = \langle \alpha_i, \alpha_j \rangle$. 证明: 向量组 $\alpha_1, \cdots, \alpha_r$ 线性无关当且仅当 $\det(A) \neq 0$.

Pf. "$\Leftarrow$" If linearly dependent, $\exists\, x \in \mathbb{R}^r \setminus \{0\}$ s.t. $\sum x_i \alpha_i = 0$.

Then $\langle \alpha_i, \sum x_j \alpha_j \rangle = \sum_j \langle \alpha_i, \alpha_j \rangle x_j = 0 \quad \forall i \in [\![1, r]\!]$.

$\Longleftrightarrow$ $Ax = 0$. Namely, x is a nontrivial solution of $AX = 0$.

$\Leftrightarrow$ $\det A = 0$.

"$\Rightarrow$" If $\det A = 0$, then $\exists\, x \in \mathbb{R}^r \setminus \{0\}$ s.t. $\langle \alpha_i, \sum x_j \alpha_j \rangle = 0$. $\forall i$.

Restricting to the subspace $\mathrm{span}_{\mathbb{R}}\{\alpha_i \mid i\}$, $\langle \cdot, \cdot \rangle$ is still a inner product. Therefore, $\sum x_j \alpha_j = 0$. $\Rightarrow$ linearly dependent.

Advanced Linear Algebra II - HW8

$\boxed{2}$

习题 7.1.18. 假设 $A \in \mathbf{M}_n(\mathbb{R})$ 既是正交矩阵又是上三角阵. 证明: A 必然是对角阵, 而且它的对角线元素均为 ± 1.

Pf. Since A is an orthogonal matrix, $A^T = A^{-1}$.

LHS is lower triangular and RHS is upper triangular.

Thus A is diagonal, say $A = \mathrm{diag}(\lambda_1, \cdots, \lambda_n)$.

Then $A^{-1} = \mathrm{diag}(\lambda_1^{-1}, \cdots, \lambda_n^{-1}) = A \Rightarrow \lambda_i = \pm 1$.

习题 7.2.2. 设 V 是 n 维内积空间.

1. 对任意非零向量 $x \in V$, 定义映射
$$\tau_x : V \longrightarrow V; \quad v \longmapsto v - 2\frac{\langle v, x\rangle}{\|x\|^2}x.$$
证明: τ_x 是个线性变换, 并且是第二类正交变换 (参见思考题 7.6). 能够写成 τ_x 这种形式的线性变换称为**镜面反射**或**镜像变换**.
2. 证明: 若 $\mathscr{A} \in \mathrm{End}(V)$ 是个镜面反射, 则 $\mathscr{A}^2 = I$.
3. 假设 $\mathscr{A} \in \mathrm{End}(V)$ 是正交变换, 1 是它的一个特征值, 并且满足 $\dim E(1, \mathscr{A}) = n - 1$. 证明: $\mathscr{A}$ 是个镜面反射.

Pf. 1. $\tau_x(mv + lw) = mv + lw - 2\langle mv + lw, x\rangle / \|x\|^2 \cdot x$

$= m\left(v - \frac{2\langle v, x\rangle}{\|x\|^2}x\right) + l\left(w - \frac{2\langle w, x\rangle}{\|x\|^2}\cdot x\right)$

$= m\,\tau_x(v) + l\,\tau_x(w)$

Let $W = \{v \in V : \langle v, x\rangle = 0\}$. $\dim W = n-1$ and $\tau_x|_W = \mathrm{id}_W \Rightarrow V = W \oplus \mathbb{R}x = W \oplus E(-1, \tau_x)$

2. $\tau_x^2(v) = \tau_x\left(v - \frac{2\langle v, x\rangle}{\|x\|^2}\cdot x\right) = v - \frac{2\langle v, x\rangle}{\|x\|^2}x - \frac{2\langle v, x\rangle}{\|x\|^2}\tau_x(x) = v$

3. $\mathscr{A}$ is orthogonal $\Rightarrow$ $\mathscr{A}$ is diagonalizable.

$\Rightarrow V = E(1, \mathscr{A}) \oplus E(-1, \mathscr{A})$

Let $x \in E(-1, \mathscr{A}) \setminus \{0\}$. $x \perp E(1, \mathscr{A})$ $\left(\forall v \in E(1, \mathscr{A}),\ \langle x, v\rangle = \langle \mathscr{A}x, \mathscr{A}v\rangle\right)$

$\Rightarrow \mathcal{A} = T_x$

习题 7.2.3. 假设 $\mathscr{A} \in \mathrm{End}(V)$ 是个镜面反射 (参见习题 7.2.2).

1. 证明映射
$$\varphi: V \times V \longrightarrow \mathbb{R}; \quad (u, v) \longmapsto \langle \mathscr{A}u, v \rangle$$
是 V 上的对称双线性型.

2. 证明: 若 $\mathscr{B} \in \mathrm{End}(V)$ 是正交变换, 则 $\mathscr{B}^{-1}\mathscr{A}\mathscr{B}$ 仍是镜面反射.

1. 双线性: 保加法数乘.

对称: $\varphi(v,u) = \langle \mathcal{A}v, u \rangle = \langle v, \mathcal{A}^{-1}u \rangle = \langle v, \mathcal{A}u \rangle = \varphi(u,v)$

2. 设 $\mathcal{A}(v) = v - \frac{2\langle v, x \rangle}{\|x\|^2} x$, 则

$$\mathcal{B}^{-1}\mathcal{A}\mathcal{B}(v) = \mathcal{B}^{-1}\left(\mathcal{B}(v) - \frac{2\langle \mathcal{B}(v), x \rangle}{\|x\|^2} x\right) = v - 2\frac{\langle \mathcal{B}(v), x \rangle}{\|x\|^2} \mathcal{B}^{-1}(x)$$
$$= v - \frac{2\langle v, \mathcal{B}^{-1}(x) \rangle}{\|\mathcal{B}(x)\|^2} \mathcal{B}^{-1}(x) = T_{\mathcal{B}^{-1}(x)}$$

习题 7.2.6. 设 $M \in \mathbf{M}_n(\mathbb{R})$. 如果存在 (在 $\mathbb{R}^n$ 标准内积意义下) 长度为 1 的列向量 $\alpha \in \mathbb{R}^n$ 使得 $M = I_n - 2\alpha\alpha^T$, 则称 M 是个**镜像矩阵**.

对于 n 维内积空间 V 上的任意线性变换 $\mathscr{A}$, 证明下列条件等价:

(i) $\mathscr{A}$ 是镜像变换 (参见习题 7.2.2).

(ii) 对于 V 的任意一组规范正交基 $\mathcal{E}$, $\mathcal{M}_{\mathcal{E}}(\mathscr{A})$ 是镜像矩阵.

(iii) 存在 V 的一组规范正交基 $\mathcal{E}$ 使得 $\mathcal{M}_{\mathcal{E}}(\mathscr{A})$ 是镜像矩阵.

Pf: (i) $\Rightarrow$ (ii). Let $\mathcal{E} = (v_1, \cdots, v_n)$ be an arbitrary orthonormal basis.

Let $\mathcal{A} = T_x$ and α be the coordinate of $\frac{x}{\|x\|}$ under $\mathcal{E}$, i.e. $\alpha_i = \langle \frac{x}{\|x\|}, v_i \rangle$

Then $T_x(v_j) = v_j - \frac{2\alpha_j}{\|x\|} x$,

$$(M_{\mathcal{E}}(\mathcal{A}))_{ij} = \langle \mathcal{A}v_j, v_i \rangle = \langle v_j - \frac{2\alpha_j}{\|x\|} x, v_i \rangle = \delta_{ij} - 2\alpha_j\alpha_i$$

$\Rightarrow M_{\mathcal{E}}(\mathcal{A}) = I_n - 2\alpha \cdot \alpha^T$

(ii) $\Rightarrow$ (iii). Obv.

(iii) $\Rightarrow$ (i). Suppose $\mathcal{E} = (v_1, \cdots, v_n)$. Let $x = \mathcal{E}\alpha = \sum_{i=1}^{n} \alpha_i v_i$.

$$\mathcal{A}(v_i) = v_i - 2\sum_{j=1}^{n} \alpha_i \alpha_j v_j = v_i - 2\alpha_i \cdot x = v_i - \frac{2\langle x, v_i \rangle}{\|x\|^2} x \Rightarrow \mathcal{A} = T_x.$$

习题 7.2.10. 设 $\mathcal{E} = (e_1, e_2, \cdots, e_n)$ 是内积空间 V 中的一组规范正交基. 定义线性变换 $\mathscr{A} \in \mathrm{End}(V)$ 为 $v \mapsto \mathscr{A}v = \langle v, e_1 \rangle e_2$ (参见例 7.2.9).

求矩阵 $\mathcal{M}_{\mathcal{E}}(\mathscr{A})$ 和 $\mathcal{M}_{\mathcal{E}}(\mathscr{A}^*)$.

$$\mathscr{A}(e_i) = \begin{cases} e_2 & i=1 \\ 0 & \text{otherwise} \end{cases}, \quad \mathcal{M}_{\mathcal{E}}(\mathscr{A}) = \begin{pmatrix} 1 & & \\ & & \\ & & \end{pmatrix} = E_{21}$$

$$\mathcal{M}_{\mathcal{E}}(\mathscr{A}^*) = E_{12}$$

习题 7.2.11. 设 $V = \mathbf{M}_n(\mathbb{R})$. 考虑 V 上的 Frobenius 内积 (参见习题 7.1.8):

$$V \times V \longrightarrow \mathbb{R}; \quad (A, B) \longmapsto \langle A, B \rangle := \mathrm{Tr}(A^T B).$$

给定矩阵 $M \in \mathbf{M}_n(\mathbb{R})$, 定义 $\varphi \in \mathrm{End}(V)$ 为 $\varphi(A) = MA$. 求 (φ 相对于 Frobenius 内积的) 伴随变换 φ^* 的表达式.

解. $\langle \varphi(A), B \rangle = \langle MA, B \rangle = \mathrm{Tr}(A^T M^T B)$

同时, $\langle A, \varphi^*(B) \rangle = \mathrm{Tr}(A^T \varphi^*(B))$

令 $A = E_{ij}$ 则 $\varphi^*(B)_{ij} = (M^T B)_{ij}$, $\forall i, j$.

故 $\varphi^*(B) = M^T B$

习题 8.1.2. 求矩阵 $A = \begin{pmatrix} 2 & \mathrm{i} \\ 1 & 0 \\ \mathrm{i} & 1 \end{pmatrix}$ 的 QR 分解.

Let $v_1 = (2, 1, i)^T$, $v_2 = (i, 0, 1)^T \in \mathbb{C}^3$.

$e_1 = \frac{v_1}{\|v_1\|} = \frac{v_1}{\sqrt{6}}$, $\quad \xi_2 = v_2 - \langle e_1, v_2 \rangle e_1 = v_2 - \frac{i}{6}(2, 1, i)^T = (\frac{2i}{3}, -\frac{i}{6}, \frac{7}{6})$

$e_2 = \frac{\xi_2}{\|\xi_2\|} = \frac{\xi_2}{\sqrt{\frac{11}{6}}} \Rightarrow v_2 = \sqrt{\frac{11}{6}}\, e_2 + \frac{i}{\sqrt{6}} e_1$

$$A = (v_1, v_2) = (e_1, e_2)\begin{bmatrix} \sqrt{6} & \frac{i}{\sqrt{6}} \\ & \sqrt{\frac{11}{6}} \end{bmatrix} = \begin{bmatrix} \frac{2}{\sqrt{6}} & \frac{4i}{\sqrt{66}} \\ \frac{1}{\sqrt{6}} & -\frac{i}{\sqrt{66}} \\ \frac{i}{\sqrt{6}} & \frac{7}{\sqrt{66}} \end{bmatrix} \cdot \begin{bmatrix} \sqrt{6} & \frac{i}{\sqrt{6}} \\ & \sqrt{\frac{11}{6}} \end{bmatrix}$$

习题 8.1.3. 设 $V = \mathbf{M}_n(\mathbb{C})$.

1. 证明: 映射

$$V \times V \longrightarrow \mathbb{C}\,; \quad (A, B) \longmapsto \langle A, B \rangle := \mathrm{Tr}\left(\overline{A}^T B\right)$$

   是 V 上的 Hermite 内积.

2. 设 U 是 V 中的对角矩阵构成的子空间. 求 U 关于以上 Hermite 内积的正交补 $U^\perp$.

证: 1. Ξ 线性: $\langle aA+bB, C\rangle = \mathrm{Tr}((\overline{aA+bB})^T C) = \bar{a}\mathrm{Tr}(\bar{A}^T C) + \bar{b}\mathrm{Tr}(\bar{B}^T C) = \bar{a}\langle A, C\rangle + \bar{b}\langle B, C\rangle$

$\langle A, bB+cC\rangle = \mathrm{Tr}(\bar{A}^T(bB+cC)) = b\mathrm{Tr}(\bar{A}^T B) + c\mathrm{Tr}(\bar{A}^T C) = b\langle A, B\rangle + c\langle A, C\rangle$

共轭对称: $\langle A, B\rangle = \mathrm{Tr}(\bar{A}^T B) = \mathrm{Tr}(B^T \bar{A}) = \overline{\mathrm{Tr}(\bar{B}^T A)} = \langle B, A\rangle$

正定: $\langle A, A\rangle = \mathrm{Tr}(\bar{A}^T A) = \sum |a_{ii}|^2 \geq 0$ 当且仅当 $A=0$ 时取"=".

2. 设 $D = \mathrm{diag}(d_1, \cdots, d_n)$, $d_i \in \mathbb{C}$. $A = (a_{ij})$

$\langle D, A\rangle = \mathrm{Tr}(\bar{D}^T A) = \sum_{i=1}^{n} \bar{d_i} a_{ii} = 0$ 对任意 $d_i \in \mathbb{C}$ 均成立.

$\Rightarrow a_{ii} = 0, \forall i$

$\Rightarrow U^\perp = \{A \in \mathrm{Mat}_n(\mathbb{C}) : A_{ii} = 0 \ \forall i\}$

习题 8.1.5. 设 V 为实内积空间或酉空间, U, W 是 V 的有限维子空间. 证明 U 和 W 正交的充分必要条件是 $P_U P_W = 0$.

证: U 与 W 正交 $\Leftrightarrow \forall (u, w) \in U \times W$, $\langle u, w\rangle = 0$

"$\Rightarrow$" $P_U P_W(v) = P_U(w) = 0$ 其中 $w' = P_W(u) \in W$

"$\Leftarrow$" $\forall w \in W$, $P_U P_W(w) = P_U(w) = 0 \Rightarrow \forall u \in U, \langle u, w\rangle = 0$

习题 8.1.7. 设 V 是实内积空间或酉空间, $W \subseteq V$ 是有限维子空间, $\mathscr{A} \in \mathrm{End}(V)$ 满足 $\langle \mathscr{A}u, \mathscr{A}v\rangle = \langle u, v\rangle$ 对所有 $u, v \in V$ 成立.

则当 W 是 $\mathscr{A}$ 的不变子空间时, $W^\perp$ 也是 $\mathscr{A}$ 的不变子空间. (提示: $\mathscr{A}|_W : W \to W$ 是满射.)

证: $\forall u \in W^\perp$, $w \in W$. $\langle \mathscr{A}w, \mathscr{A}w\rangle = \langle w, w\rangle \geq 0 \Rightarrow \mathscr{A}|_W$ 满

假设 $w' \in W$ 满 $\mathscr{A}w' = w$, 则

$\langle \mathscr{A}u, w\rangle = \langle u, w'\rangle = 0. \Rightarrow W^\perp$ 是 $\mathscr{A}$-不变的.

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#

习题 7.2.4. 设 u, v 是内积空间 V 中两个不同的向量, 且 $\|u\| = \|v\|$. 证明: 存在一个镜面反射 (参见习题 7.2.2) $\mathscr{A}$ 使得 $\mathscr{A}u = v$.

Let $\mathscr{A} = T_{u-v}$.

$$\mathscr{A}u = T_{u-v}(u) = u - \frac{2\langle u-v, u\rangle}{\|u-v\|}(u-v)$$

$$= \left(1 - \frac{2\langle u-v, u\rangle}{\|u-v\|^2}\right)u + \frac{2\langle u-v, u\rangle}{\|u-v\|^2} v$$

Note that
$$\begin{aligned} 2\langle u-v, u\rangle &= 2\langle u, u\rangle - 2\langle v, u\rangle \\ &= \langle u, u\rangle - \langle v, u\rangle - \langle u, v\rangle + \langle v, v\rangle \\ &= \langle u-v, u-v\rangle \\ &= \|u-v\|^2 \end{aligned}$$

Thus, $\mathscr{A}u = v$.

习题 7.2.13. 对下列对称阵 A, 分别求一个正交矩阵 Q 使 $Q^{-1}AQ$ 成为对角阵:

$$A = \begin{pmatrix} 2 & -2 & 0 \\ -2 & 1 & -2 \\ 0 & -2 & 0 \end{pmatrix};$$

Step I. Find P s.t. $P^{-1}AP = \text{diag}$.

$$|\lambda I - A| = \begin{vmatrix} \lambda-2 & 2 & 0 \\ 2 & \lambda-1 & 2 \\ 0 & 2 & \lambda \end{vmatrix} = \lambda(\lambda-2)(\lambda-1) - 4\lambda + 8 - 4\lambda$$

$$= (\lambda-1)(\lambda^2-2\lambda-8) = (\lambda+2)(\lambda-1)(\lambda-4)$$

$\lambda = -2$: Let $(A+2I)\xi = \begin{pmatrix} 4 & -2 & 0 \\ -2 & 3 & -2 \\ 0 & -2 & 2 \end{pmatrix}\xi = 0 \Rightarrow \xi_1 = (1, 2, 2)^T$

$\lambda = 1$: Let $(A-I)\xi = \begin{pmatrix} 1 & -2 & 0 \\ -2 & 0 & -2 \\ 0 & -2 & -1 \end{pmatrix}\xi = 0 \Rightarrow \xi_2 = (2, 1, -2)^T$

$\lambda=4$: let $(A-4I)\xi = \begin{pmatrix} -2 & -2 & 0 \\ -2 & -3 & -2 \\ 0 & -2 & -4 \end{pmatrix}\xi = 0 \Rightarrow \xi_3=(2,-2,1)^T$

Step II. Orthogonalize P

let $e_1 = \frac{\xi_1}{\|\xi_1\|} = \frac{1}{3}(1,2,2)^T$

let $v_2 = \xi_2 - \langle \xi_2, e_1\rangle e_1$

$= (2,1,-2)^T$

$e_2 = \frac{v_2}{\|v_2\|} = \frac{1}{3}(2,1,-2)^T$

let $v_3 = \xi_3 - \langle \xi_3, e_1\rangle e_1 - \langle \xi_2, e_2\rangle e_2$

$= (2,-2,1)^T$

$e_3 = \frac{v_3}{\|v_3\|} = \frac{1}{3}(2,-2,1)^T$

Then let $Q = (e_1\, e_2\, e_3) = \frac{1}{3}\begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{pmatrix}$

$$Q^{-1}AQ = \begin{pmatrix} -2 & & \\ & 1 & \\ & & 4 \end{pmatrix}$$

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}.$$

$$|\lambda I - A| = (\lambda-4)\begin{vmatrix} 1 & 1 & 1 & 1 \\ & \lambda & & \\ & & \lambda & \\ & & & \lambda \end{vmatrix} = \lambda^3(\lambda-4)$$

$\lambda=0$: $A\xi=0 \Rightarrow \xi_1=(-1,1,0,0)^T$, $\xi_2=(-1,0,1,0)^T$, $\xi_3=(-1,0,0,1)^T$

$\lambda=4$: $A\xi=4\xi \Rightarrow \xi_4=(1,1,1,1)^T$

Let $e_1 = \frac{g_1}{\|g_1\|} = \frac{1}{\sqrt{2}}(-1, 1, 0, 0)^T$

Let $v_2 = g_2 - \langle g_2, e_1 \rangle e_1 = g_2 - \frac{1}{2} g_1 = (-\frac{1}{2}, -\frac{1}{2}, 1, 0)^T$

$e_2 = \frac{v_2}{\|v_2\|} = \frac{1}{\sqrt{6}}(-1, -1, 2, 0)$

Let $v_3 = g_3 - \langle g_3, e_2 \rangle e_2 - \langle g_3, e_1 \rangle e_1$

$= (-1, 0, 0, 1)^T - \frac{1}{6}(-1, -1, 2, 0)^T - \frac{1}{2}(-1, 1, 0, 0)^T$

$= (-\frac{1}{3}, -\frac{1}{3}, -\frac{1}{3}, 1)^T$

$e_3 = \frac{v_3}{\|v_3\|} = \frac{\sqrt{3}}{6}(-1, -1, -1, 3)^T$

Let $v_4 = g_4 \quad (g_4 \perp e_i,\ i = 1, 2, 3)$

$e_4 = \frac{v_4}{\|v_4\|} = \frac{1}{2}(1, 1, 1, 1)^T$

$$Q = \begin{pmatrix} -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} & -\frac{\sqrt{3}}{6} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{2}{\sqrt{6}} & -\frac{\sqrt{3}}{6} & \frac{1}{\sqrt{2}} \\ 0 & -\frac{1}{\sqrt{6}} & -\frac{\sqrt{3}}{6} & \frac{1}{\sqrt{2}} \\ 0 & 0 & \frac{\sqrt{3}}{2} & \frac{1}{\sqrt{2}} \end{pmatrix}$$

习题 8.2.2. 将一个复方阵 $A \in \mathbf{M}_n(\mathbb{C})$ 写成 $A = P + \mathrm{i}Q$, 其中 $P, Q \in \mathbf{M}_n(\mathbb{R})$. 证明: A 是酉矩阵当且仅当 P^TQ 是对称阵且 $P^TP + Q^TQ = I_n$.

证: A 是酉阵 $\Leftrightarrow \bar{A}^T A = I_n \Leftrightarrow P^TP + Q^TQ + i(P^TQ - Q^TP) = I_n$

$$\Leftrightarrow \begin{cases} P^TP + Q^TQ = I_n \\ P^TQ = Q^TP = (P^TQ)^T \end{cases}$$

习题 8.2.4. 证明任意一个二阶酉矩阵 U 可以分解为

$$U=\begin{pmatrix} e^{i\theta_1} & 0 \\ 0 & e^{i\theta_2}\end{pmatrix}\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta\end{pmatrix}\begin{pmatrix} e^{i\theta_3} & 0 \\ 0 & e^{i\theta_4}\end{pmatrix},\quad \text{其中 } \theta,\theta_i\in\mathbb{R}.$$

设 $U=(u_{ij})$，$u_{ij}\in\mathbb{C}$，$i,j\in\{1,2\}$. 则

$$U^*U=\begin{pmatrix} |u_{11}|^2+|u_{21}|^2 & \overline{u_{11}}u_{12}+\overline{u_{21}}u_{22} \\ \overline{u_{12}}u_{11}+\overline{u_{22}}u_{21} & |u_{12}|^2+|u_{22}|^2\end{pmatrix}=I_2$$

令 $u_{11}=e^{i\alpha}\cdot\cos\theta$，$u_{21}=e^{i\beta}\sin\theta$ $\alpha,\beta,\theta\in\mathbb{R}$

$u_{12}=e^{i\gamma}\cos\xi$，$u_{22}=e^{i\omega}\cdot\sin\xi$，$\gamma,\omega,\xi\in\mathbb{R}$

则 $\overline{u_{11}}u_{12}+\overline{u_{21}}u_{22}=e^{i(\gamma-\alpha)}\cos\xi\cos\theta+e^{i(\omega-\beta)}\sin\xi\sin\theta=0$.

$\Rightarrow$ "$(\gamma-\alpha)-(\omega-\beta)\in 2k\pi, k\in\mathbb{Z}$. 且 $\cos(\xi-\theta)=0$"

或 "$(\gamma-\alpha)-(\omega-\beta)\in 2k\pi+\pi, k\in\mathbb{Z}$. 且 $\cos(\xi+\theta)=0$"

不妨令 $\xi-\theta=\frac{\pi}{2}$，$\gamma-\alpha=\omega-\beta$，则 $u_{12}=-e^{i\gamma}\sin\theta$，$u_{22}=e^{i\omega}\cos\theta$

$$U=\begin{pmatrix} e^{i\alpha}\cos\theta & -e^{i\gamma}\sin\theta \\ e^{i\beta}\sin\theta & e^{i\omega}\cos\theta\end{pmatrix}\overset{令}{=}\begin{pmatrix} e^{i\theta_1} & \\ & e^{i\theta_2}\end{pmatrix}\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta\end{pmatrix}\begin{pmatrix} e^{i\theta_3} & \\ & e^{i\theta_4}\end{pmatrix}$$

用待定系数法可解得 θ_i.

不妨取，
$$\begin{cases}\theta_1+\theta_3=\alpha\\ \theta_1+\theta_4=\gamma\\ \theta_2+\theta_3=\beta\\ \theta_2+\theta_4=\omega\end{cases}$$
由于 $\gamma+\beta=\omega+\alpha$ 方程一定有解.

习题 8.2.5. 证明 Schur 不等式: 设 $\lambda_1,\cdots,\lambda_n$ 是复方阵 $A\in\mathbf{M}_n(\mathbb{C})$ (在重数计入意义下) 的所有特征值. 则

$$\operatorname{Tr}(A\overline{A}^T)\ge\sum_{i=1}^n|\lambda_i|^2,$$

并且等号成立的充分必要条件是 A 为正规矩阵. (提示: 使用 Schur 定理.)

证：设 U 为酉方阵 st. $A=U^{-1}QU$，其中 Q 为上三角.

$\operatorname{Tr}(AA^*)=\operatorname{Tr}(U^{-1}QUU^*Q^*U)=\operatorname{Tr}(U^{-1}QQ^*U)=\operatorname{Tr}(QQ^*UU^{-1})$

$=\operatorname{Tr}(QQ^*)=\sum_{i=1}^n\|\alpha_i\|^2\ge\sum|\lambda_i|^2$，其中 α_i 为 Q 的第 i 列. λ_i 为对角元.

取"="iff $\alpha_i = \begin{pmatrix} 0 \\ \vdots \\ \lambda_i \\ \vdots \\ 0 \end{pmatrix}$. 即 A 为正规阵.

习题 8.2.6. 设 V 是有限维酉空间, $\mathscr{A} \in \mathrm{End}(V)$. 证明: $\mathscr{A}$ 是正规变换的充分必要条件是 $\mathscr{A}$ 的特征向量都是伴随变换 $\mathscr{A}^*$ 的特征向量. (提示: 使用 Schur 定理.)

By definition, $\mathscr{A}^*\mathscr{A} = \mathscr{A}\mathscr{A}^*$. Thus eigenvectors of $\mathscr{A}$ are e'vect of $\mathscr{A}^*$.

Conversely, let $T = (t_{ij}) := M_{\mathcal{E}}(\mathscr{A})$ be an upper triangular matrix under $\mathcal{E} = (e_1, \cdots, e_n)$. In this case, $\overline{T}^T = M_{\mathcal{E}}(\mathscr{A}^*)$.

Then $\mathscr{A}e_1 = t_{11}e_1 \Rightarrow e_1$ is an eigenvector of $\mathscr{A}$, and thus of $\mathscr{A}^*$.

Since $\mathscr{A}^*(e_1) = \bar{t}_{11}e_1 + \bar{t}_{12}e_2 + \cdots + \bar{t}_{1n}e_n$, $t_{12} = \cdots = t_{1n} = 0$.

Let $W = \mathrm{span}(e_2, \cdots, e_n)$. $\mathscr{A}|_W$ has the same eigenvectors of $\mathscr{A}^*|_W$.

Then by induction on dim, we can prove

$$M_{\mathcal{E}}(\mathscr{A}) \text{ is a diagonal matrix.}$$

Thus $\mathscr{A}\mathscr{A}^* = \mathscr{A}^*\mathscr{A}$.

习题 8.2.7. 设 V 是有限维酉空间, $\dim V \geq 1$. 设 $S \subseteq \mathrm{End}(V)$ 是由 V 上的自伴算子构成的子集. 证明: S 一定不是 $\mathrm{End}(V)$ 作为复向量空间的线性子空间.

Let σ be a nonzero self-adjoint operator.

$(i\sigma)^* = -i\sigma^* = -i\sigma$. Thus S does NOT preserve scalar products.

习题 8.2.8. 设 $\mathscr{A}, \mathscr{B} \in \mathrm{End}(V)$ 均为自伴算子. 证明: $\mathscr{A}\mathscr{B}$ 是自伴算子的充分必要条件是 $\mathscr{A}\mathscr{B} = \mathscr{B}\mathscr{A}$.

习题 8.2.9. 设 $\mathscr{A}, \mathscr{B} \in \mathrm{End}(V)$ 均为自伴算子. 证明: $\mathscr{A}\mathscr{B} + \mathscr{B}\mathscr{A}$ 和 $\mathrm{i}(\mathscr{A}\mathscr{B} - \mathscr{B}\mathscr{A})$ 都是自伴算子.

习题 8.2.10. 对任意 $\mathscr{A} \in \mathrm{End}(V)$ 证明: 如果 $\mathscr{A}$ 满足以下三个条件中的两个, 则 $\mathscr{A}$ 也满足第三个条件: (i) $\mathscr{A}$ 自伴; (ii) $\mathscr{A}$ 是酉变换; (iii) $\mathscr{A}^2 = I$.

8. $(\mathscr{A}\mathscr{B})^* = \mathscr{B}^*\mathscr{A}^* = \mathscr{B}\mathscr{A}$

$\mathscr{A}\mathscr{B}$ self-adjoint $\Leftrightarrow$ $\mathscr{B}\mathscr{A} = \mathscr{A}\mathscr{B}$

9. $(\mathscr{A}\mathscr{B} + \mathscr{B}\mathscr{A})^* = \mathscr{B}^*\mathscr{A}^* + \mathscr{A}^*\mathscr{B}^* = \mathscr{A}\mathscr{B} + \mathscr{B}\mathscr{A}$

$(i\mathscr{A}\mathscr{B} - i\mathscr{B}\mathscr{A})^* = -i\mathscr{B}^*\mathscr{A}^* + i\mathscr{A}^*\mathscr{B}^* = i\mathscr{A}\mathscr{B} - i\mathscr{B}\mathscr{A}$

10. i)+ii) $\Rightarrow$ iii)

$\mathscr{A} = \mathscr{A}^*$, $\mathscr{A}\mathscr{A}^* = I_n \Rightarrow \mathscr{A}^2 = I_n$

i)+iii) $\Rightarrow$ ii)

$\mathscr{A}\mathscr{A}^* = \mathscr{A}^2 = I_n$

ii)+iii) $\Rightarrow$ i)

$\mathscr{A}^2 = I_n \Rightarrow \mathscr{A}$ is invertible and $\mathscr{A}^{-1} = \mathscr{A}$.

$\Rightarrow \mathscr{A}^* = \mathscr{A}^{-1} = \mathscr{A}$.

Advanced Linear Algebra II – HW10

卦

习题 7.2.19. 设 $\mathscr{A} \in \mathrm{End}(V)$ 为正规变换. 证明: $\mathrm{Im}(\mathscr{A}) = \mathrm{Im}(\mathscr{A}^*)$.

Since $\mathrm{Im}(\mathscr{A}) = \ker(\mathscr{A}^*)^{\perp}$ and $\mathrm{Im}(\mathscr{A}^*) = \ker(\mathscr{A})^{\perp}$, it suffices to show $\ker \mathscr{A}^* = \ker \mathscr{A}$.

Suppose $\mathscr{A}^* v = 0$. Then $\|\mathscr{A} v\|^2 = \langle v, \mathscr{A}^*\mathscr{A} v\rangle = \langle v, \mathscr{A}\mathscr{A}^* v\rangle = 0$.
$\Rightarrow \mathscr{A} v = 0$. $\Rightarrow \ker \mathscr{A}^* \subseteq \ker \mathscr{A}$. By the symmetry, we are done.

习题 7.2.21. 设 $A \in \mathbf{M}_n(\mathbb{R})$ 为对称矩阵. 对于每个 $i \in [\![1, n]\!]$, 记 c_i 为 A 的所有 i 阶主子式之和. 证明: 如果 $c_i \geq 0$ 对每个 $i \in [\![1, n]\!]$ 成立, 那么 A 是半正定的. 如果再有 $c_n > 0$, 则 A 是正定的.

Pf. By spectral theorem, A is orthogonally similar to a diagonal matrix $D = \mathrm{diag}(\lambda_1, \cdots, \lambda_n)$ with $\lambda_i \in \mathbb{R}$, say $A = \bar{Q}^T D Q$

On one hand, $|\lambda I - A| = \sum_{k=0}^{n} (-1)^k c_k \lambda^{n-k}$ (Let $c_0 = 1$) $\cdots$ ①

On the other hand,

$$|\lambda I - A| = |\lambda I - D| = \prod_{i=1}^{n} (\lambda - \lambda_i) \quad \cdots ②$$

Suppose A is NOT positive semi-definite, i.e., $\exists\, \lambda_i < 0$.

Then taking $\lambda = \lambda_i$ gives $|\lambda_i I - A| = 0$ by ② and

$$|\lambda_i I - A| = \sum_{k=0}^{n} (-1)^k c_k \lambda_i^{n-k} = (-1)^n \left(\sum_{k=0}^{n} c_k (-\lambda_i)^{n-k} \right) \text{ by ①}$$

with $(-\lambda_i)^{n-k} > 0$. Then $c_k = 0$ for all k, contrary to $c_0 = 1$

Suppose A is NOT positive definite, i.e., $\exists\ \lambda_i \leq 0$.

Then taking $\lambda = \lambda_i$ gives $|\lambda_i I - A| = 0$ by ② and

$$|\lambda_i I - A| = (-1)^n \sum_{k=0}^{n} C_k (-\lambda_i)^{n-k} \quad \text{by ①}$$

where $(-\lambda_i)^{n-k} \geq 0$.

Then $-\lambda_i = 0$, which implies $C_n = (-1)^n \lambda_1 \lambda_2 \cdots \lambda_n = 0$, contrary to our assumption $(C_i > 0)$.

习题 7.2.22. 设 $A, B \in \mathbf{M}_n(\mathbb{R})$ 均为对称阵, 其中 A 正定. 证明: 存在可逆矩阵 $P \in \mathbf{M}_n(\mathbb{R})$ 使得 $P^T AP$ 和 $P^T BP$ 同时为对角阵. (提示: 先考虑 $A = I_n$ 的情况.)

证. $\exists$ 正交阵 Q' st. $Q'^T A Q' = \mathrm{diag}(\lambda_1, \cdots, \lambda_n)$. $\Rightarrow \exists$ 对称阵 $Q^T A Q = I_n$

故 $\exists P$ 正交 st. $P^{-1} Q^T B Q P$ 对角, 此时 $P^{-1} Q^T A Q P = I_n$

$\Rightarrow QP$ 满足条件.

习题 7.2.24. 设 $b, c \in \mathbb{R}$ 且 $b^2 - 4c < 0$. 举例说明: 存在线性变换 $\mathscr{A} \in \mathrm{End}(V)$ 使得 $\mathscr{A}^2 + b\mathscr{A} + cI$ 不可逆.

(这说明: 引理 7.2.24 如果不假设 $\mathscr{A}$ 自伴, 那么结论不再成立.)

$c = -b = 1$, $V = \mathbb{R}^2$, $\mathcal{E}$ 标准基, $\mathcal{M}_{\mathcal{E}}(\mathscr{A}) = \begin{pmatrix} 1 & \\ & 1 \end{pmatrix}$

$\mathcal{M}_{\mathcal{E}}(\mathscr{A}^2 + b\mathscr{A} + cI) = 0$.

习题 7.2.25. 设 $\mathscr{A}, \mathscr{B}$ 是 n 维内积空间 V 上的两个自伴算子. 证明下列陈述等价:

(i) 存在 V 的一组规范正交基 $\mathcal{E}$ 使得 $\mathcal{M}_{\mathcal{E}}(\mathscr{A})$ 和 $\mathcal{M}_{\mathcal{E}}(\mathscr{B})$ 同时为对角阵.

(ii) $\mathscr{A}$ 和 $\mathscr{B}$ 可交换.

Denote by $A = \mathcal{M}_{\mathcal{E}}(\mathscr{A})$, $B = \mathcal{M}_{\mathcal{E}}(\mathscr{B})$

i) $\Rightarrow$ ii) Suppose $P^T A P = D_1$, $P^T B P = D_2$ where D_1 and D_2 are diagonal matrices. Then D_1 commutes with D_2.

Thus $AB = PD_1P^TPD_2P^T = PD_1D_2P^T = PD_2D_1P^T$

$= PD_2P^TPD_1P^T = BA$

ii) $\Rightarrow$ i) Since $\mathscr{A}$ and $\mathscr{B}$ are self-adjoint, they are diagonalizable.

Let $v \in E(\lambda, \mathscr{A})$, then $\mathscr{A}\mathscr{B}v = \mathscr{B}\mathscr{A}v = \lambda \mathscr{B}v$

Thus $E(\lambda, \mathscr{A})$ is $\mathscr{B}$-invariant.

In each $E(\lambda, \mathscr{A})$, we can find an orthonormal basis $\varepsilon_1^{(\lambda)}, \cdots \varepsilon_k^{(\lambda)}$ diagonalizing $\mathscr{B}|_{E(\lambda,\mathscr{A})}$. Combining all of them gives an orthonormal basis of V which diagonalizes $\mathscr{A}$ and $\mathscr{B}$ simultaneously.

习题 7.2.27. 设 A 为 n 阶实正规矩阵. 假设 $\lambda \in \mathbb{C}$ 是 A 的一个复特征值, 列向量 $\alpha \in \mathbb{C}^n$ 满足 $A\alpha = \lambda\alpha$. 证明: $A^T\alpha = \bar{\lambda}\alpha$.

Idea: show $\|(A^T - \bar{\lambda} I_n)\alpha\| = ((A^T - \bar{\lambda} I_n)\alpha)^* ((A^T - \bar{\lambda} I_n)\alpha) = 0$

Since $(A^T - \bar{\lambda} I_n)^* = A - \lambda I_n$,

$$\|(A^T - \bar{\lambda} I_n)\alpha\| = \alpha^* (A - \lambda I_n)(A^T - \bar{\lambda} I_n)\alpha = \alpha^* (AA^T - \lambda A^T - \bar{\lambda} A + \|\lambda\|^2 I_n)\alpha$$

$$= \underbrace{\alpha^* (A^TA\alpha) - \lambda \alpha^* A^T \alpha}_{①} \underbrace{- \bar{\lambda} \alpha^* A \alpha + \|\lambda\|^2 \alpha^* \alpha}_{②}$$

$$= 0$$

习题 7.2.28. 设 $\mathscr{A} \in \mathrm{End}(V)$ 是正规变换且最小多项式为 $g(X) = (X-a)^2 + b^2$, 其中 $a, b \in \mathbb{R}$, $b \neq 0$. 证明: $\mathscr{A}$ 是可逆变换, 且 $\mathscr{A}^* = (a^2+b^2)\mathscr{A}^{-1}$.

Pf. By the spectral theorem, $\exists$ orthonormal basis $\mathcal{E}$ st.

$$\mathcal{M}_{\mathcal{E}}(\mathscr{A}) = \begin{pmatrix} A_1 & & \\ & A_1 & \\ & & \ddots \\ & & & A_1 \end{pmatrix}, \quad A_1 = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}.$$

Since A_1 is invertible, so is $\mathcal{M}_{\mathcal{E}}(\mathscr{A})$ and $\mathscr{A}$.

$$A_1^{-1} = \frac{1}{a^2+b^2} \begin{pmatrix} a & b \\ -b & a \end{pmatrix} = \frac{A_1^T}{a^2+b^2} \Rightarrow \mathscr{A}^* = (a^2+b^2)\mathscr{A}^{-1}$$

习题 7.2.30. 设 $\mathcal{C}[-\pi,\pi]$ 是闭区间 $[-\pi,\pi]$ 上所有的实值连续函数构成的空间，在其上定义内积如下:

$$\langle f, g\rangle := \int_{-\pi}^{\pi} f(x)g(x)\mathrm{d}x.$$

取

$$V = \mathrm{span}(1, \cos(x), \cos(2x), \cos(3x), \sin(x), \sin(2x), \sin(3x)).$$

1. 证明: 对于任意 $f \in V$, 其导函数 f' 也属于 V.

2. 定义 V 上的线性变换 $\mathscr{A} : V \to V$, $f \mapsto f'$. 证明 $\mathscr{A}$ 是斜对称变换.

3. 求 V 的一组规范正交基 $\mathcal{E}$, 使得矩阵 $\mathcal{M}_{\mathcal{E}}(\mathscr{A})$ 具有正交相似标准形 (即定理 7.2.35 的结论 (iii), (iv) 或 (v) 中的形式).

证：1. 考虑基的导函数即得.

2. 令 $\mathcal{E} = \left(\frac{1}{\sqrt{2\pi}}, \frac{\cos x}{\sqrt{\pi}}, \frac{\sin x}{\sqrt{\pi}}, \cdots, \frac{\sin 3x}{\sqrt{\pi}}, \frac{\cos 3x}{\sqrt{\pi}}\right)$ 为一组基.（规范正交）

$$\mathcal{M}_{\mathcal{E}}(\mathscr{A}) = \begin{pmatrix} 0 & & & & & & \\ & & 1 & & & & \\ & -1 & & & & & \\ & & & & 2 & & \\ & & & -2 & & & \\ & & & & & & 3 \\ & & & & & -3 & \end{pmatrix}$$

反对称

3. 即 2.

习题 7.2.31. 设 $A, B \in \mathbf{M}_n(\mathbb{R})$, 其中 A 正定. 证明:

1. 对任意正整数 k, 矩阵 A^k 也正定.

2. 如果存在正整数 r 使得 B 与 A^r 可交换, 则 B 和 A 也可交换.

(提示: 利用正交相似标准形.)

Pf: 1. Suppose $A = P^T D P$, $D = \mathrm{diag}(\lambda_1, \cdots, \lambda_n)$ with $\lambda_i > 0$. and P is an orthogonal matrix.

Then $A^k = P^T \mathrm{diag}(\lambda_1^k, \cdots, \lambda_n^k) P$ is positive definite.

Method 1.

2. Since $\lambda_i > 0$, $\lambda_i^k = \lambda_j^k \Leftrightarrow \lambda_i = \lambda_j$.

We can use Lagrange interpolation to obtain a polynomial $f(t)$ s.t. $f(\lambda_i^r) = \lambda_i$. Thus $A = f(A^r)$ is a polynomial of A^r.

Then B commutes with A.

Method 2.

2. Let $C = P^T B P$. Then

B commutes with $A^k \iff CD^k = D^k C$

$\iff c_{ij}\lambda_j^k = c_{ij}\lambda_i^k, \forall i,j.$

$\iff (\lambda_j^k - \lambda_i^k) c_{ij} = 0, \forall i,j$

since $\lambda_i > 0, \forall i$ $\iff$ $\lambda_j = \lambda_i$ or $c_{ij} = 0$ $\quad \forall i,j$

$\iff (\lambda_j - \lambda_i) c_{ij} = 0, \forall i,j$

$\iff CD = DC.$

$\iff BA = AB.$

习题 7.2.32. 设 U 和 V 分别为 n 维和 m 维的内积空间, $W = \mathrm{Hom}(U, V)$. 取定 U 的一组规范正交基 $\varepsilon_1, \cdots, \varepsilon_n$.

1. 证明: 映射

$$\langle \cdot, \cdot \rangle_W : W \times W \longrightarrow \mathbb{R}; \quad (f, g) \longmapsto \sum_{i=1}^n \langle f(\varepsilon_i), g(\varepsilon_i) \rangle_V$$

是 W 上的一个内积.

2. 按照上一小题定义的内积将 W 视为内积空间. 对任意 $\mathscr{A} \in \mathrm{End}(U)$, 定义映射

$$T(\mathscr{A}) : W \longrightarrow W; \quad f \longmapsto \big(T(\mathscr{A})(f) : x \in U \longmapsto f(\mathscr{A}x) \in V\big).$$

证明: $T(\mathscr{A})$ 是 W 上的线性变换.

再证明: $T(\mathscr{A})$ 是 W 上的正交变换当且仅当 $\mathscr{A}$ 是 U 上的正交变换.

1. Routine!

2. $T(A)(f+g): x \mapsto (f+g)(Ax) = f(Ax) + g(Ax)$

$T(A)(f) + T(A)(g): x \mapsto f(Ax) + g(Ax)$

$T(A)(kf): x \mapsto f(kA)x = kf(A)x$

$kT(A)(f): x \mapsto kf(A)x$

Above holds for all $f \in W$, Thus $T(A)$ is linear.

3. $\langle T(A)(f), T(A)(g) \rangle_W = \sum \langle T(A)f(\varepsilon_i), T(A)g(\varepsilon_i) \rangle_V$

$= \sum \langle f(A\varepsilon_i), g(A\varepsilon_i) \rangle_V$

Suppose $A\varepsilon_i = \sum_j a_{ji}\varepsilon_j$. Then $f(A\varepsilon_i) = \sum_j a_{ji} f(\varepsilon_j)$

If A is orthogonal, i.e., $\sum_k a_{ik} a_{jk} = \delta_{ij}$ $\forall i, j$, then

$\langle T(A)(f), T(A)(g) \rangle_W = \sum_i \langle \sum_j a_{ji} f(\varepsilon_j), \sum_l a_{li} g(\varepsilon_l) \rangle_V$

$= \sum_{j,l} \left(\sum_i a_{ji} a_{li} \right) \langle f(\varepsilon_j), g(\varepsilon_l) \rangle_V$

$= \sum_{j,l} \delta_{jl} \langle f(\varepsilon_j), g(\varepsilon_l) \rangle_V = \langle f, g \rangle_W$

$\Rightarrow T(A)$ is orthogonal.

If $T(\mathcal{A})$ is orthogonal, we will prove $M_\mathcal{E}(\mathcal{A})M_\mathcal{E}(\mathcal{A})^T = I_n$

Let f be the projection to $\mathcal{E}_j$ and $g: \sum b_i \mathcal{E}_i \mapsto b_\ell \mathcal{E}_j$

Then $\langle T(\mathcal{A})f, T(\mathcal{A})g \rangle_W = \sum_i \langle f(\mathcal{A}\mathcal{E}_i), g(\mathcal{A}\mathcal{E}_i) \rangle_V$

$$= \sum_i \langle a_{ji}\mathcal{E}_j, a_{\ell i}\mathcal{E}_j \rangle_V$$

$$= \sum_i a_{ji}a_{\ell i} \langle \mathcal{E}_j, \mathcal{E}_j \rangle_V = \sum_i a_{ji}a_{\ell i}$$

and $\langle f, g \rangle_W = \sum_i \langle f(\mathcal{E}_i), g(\mathcal{E}_i) \rangle_V = \delta_{j\ell}$.

$\Rightarrow \sum_i a_{ji}a_{\ell i} = \delta_{j\ell} \Rightarrow \mathcal{A}$ is orthogonal.

Advanced Linear Algebra II - HW 11

甘

习题 8.3.1. 设 U 是 V 的子空间. 证明正交投影 $P_U \in \mathrm{End}(V)$ 是半正定算子.

证：设 $v = u_1 + u_2$，其中 $u_1 \in U$，$u_2 \in U^{\perp}$

$\langle P_U(v), v \rangle = \langle u_1, u_1 \rangle + \langle u_1, u_2 \rangle = \langle u_1, u_1 \rangle \geq 0$

且 $\forall v \in U^{\perp}$，$\langle P_U(v), v \rangle = 0$.

故 P_U 为半正定算子.

习题 8.3.2. 设 V 是有限维酉空间, $\mathscr{A} \in \mathrm{End}(V)$ 是正规变换. 证明 $\mathscr{A}$ 必有平方根, 即, 存在 $\mathscr{B} \in \mathrm{End}(V)$ 使得 $\mathscr{A} = \mathscr{B}^2$.

证：由谱定理，$\exists$一组规范正交基 $\varepsilon = \{e_1, \cdots, e_n\}$，$e_i$ 均为特征向量.

令 $\mathscr{A} e_i = \lambda_i e_i$. 考虑映射 $\mathscr{B} : e_i \mapsto \lambda_i^{\frac{1}{2}} e_i$，$\lambda_i^{\frac{1}{2}}$ 取任一 λ_i 的平方根均可.

则 $\mathscr{B}^2 = \mathscr{A}$.

习题 8.3.3. 假设 $\dim V > 1$. 证明恒等算子 $I \in \mathrm{End}(V)$ 有无穷多个平方根.

If $\dim V > 1$, there are infinitely many noncollinear vectors

($W = \{ v_1 + a v_2 : a \in \mathbb{K} \}$ is a set of noncollinear vectors).

Consider the reflections T_x, $x \in W$. They are pairwise distinct linear operators satisfying $T_x^2 = id$.

习题 8.3.4. 设 W 是有限维复向量空间, $\mathscr{A} \in \mathrm{End}(W)$ 是可逆线性变换. 证明 $\mathscr{A}$ 必有平方根. (提示: 利用 Jordan 标准形或 Jordan–Chevalley 分解.)

证: 令 $D_k=\sum_{i=1}^{n-k} E_{i,i+k}$, $k=0,1,\cdots,n-1$. 注意到 $D_i\cdot D_j=D_{i+j}$, ($i\geq n$ 时, $D_i=0$)

考虑方程 $(a_0D_0+a_1D_1+\cdots+a_{n-1}D_{n-1})^2=\lambda D_0+D_1=J_n(\lambda)$, $\lambda\neq 0$.

则有 $a_0^2=\lambda$, $a_0a_1+a_1a_0=1$, $a_0a_2+a_1^2+a_2a_0=0$, $\cdots$, $a_0a_{n-1}+\cdots+a_{n-1}a_0=0$

即 $a_0^2=\lambda$, $2a_0a_1=1$, $\forall\, 2\leq i\leq n-1$, $\sum_{j=0}^{i}a_ja_{i-j}=0$.

共 n 个方程, n 个未知量, 且 $a_0\neq 0$. 故一定有解.

故 Jordan 块必有平方根. 因此, $\mathscr{A}$ 一定有平方根.

习题 8.3.5. 证明或举出反例: 设 V 是有限维酉空间或实内积空间, $\mathscr{A} \in \mathrm{End}(V)$ 是自伴算子, $(e_1,\cdots,e_n)$ 是 V 的一组规范正交基. 如果对每个 $i\in[\![1,n]\!]$ 均有 $\langle \mathscr{A}e_i, e_i\rangle$ 为非负实数, 那么 $\mathscr{A}$ 是半正定算子.

证: 考虑 $V=\mathbb{R}^2$ (或 $\mathbb{C}^2$), 在 $\varepsilon=(e_1,e_2)$ 下, $M_\varepsilon(\mathscr{A})=\begin{pmatrix}1&2\\2&1\end{pmatrix}$

令 $v=e_1-2e_2$, $\mathscr{A}v=-3e_1$. $\langle\mathscr{A}v,v\rangle=-3\langle e_1,e_1-2e_2\rangle=-3<0$

故题目命题非真.

$$\mathscr{A}^*\mathscr{A}(v)=\begin{cases}0 & \langle v,u\rangle=0\\ \langle x,x\rangle\langle u,u\rangle u, & v=u\end{cases}$$

$$\mathscr{B}v=c\langle u,v\rangle u$$

习题 8.3.6. 取定非零向量 $u, x\in V$. 定义线性变换

$$\mathscr{A}: V\longrightarrow V;\quad v\longmapsto \mathscr{A}v:=\langle u,v\rangle x.$$

求 $\sqrt{\mathscr{A}^*\mathscr{A}}$ 的表达式.

$\langle\mathscr{A}^*(v),w\rangle=\langle v,\mathscr{A}w\rangle=\langle u,w\rangle\langle v,x\rangle=\langle\overline{\langle v,x\rangle}u,w\rangle$

由伴随的唯一性, $\mathscr{A}^*: v\mapsto\overline{\langle v,x\rangle}u$.

$\mathscr{A}^*\mathscr{A}: v\longmapsto\langle u,v\rangle\langle x,x\rangle u$

令 $\mathscr{B}: v\mapsto\sqrt{\frac{\langle x,x\rangle}{\langle u,u\rangle}}\langle u,v\rangle u$, 则 $\forall v\in V$.

$$\mathscr{B}^2(v)=\sqrt{\frac{\langle x,x\rangle}{\langle u,u\rangle}}\cdot\langle u,\sqrt{\frac{\langle x,x\rangle}{\langle u,u\rangle}}\langle u,v\rangle u\rangle u$$

$$=\frac{\langle x,x\rangle}{\langle u,u\rangle}\cdot\langle u,u\rangle\langle u,v\rangle u=\langle x,x\rangle\langle u,v\rangle u=\mathscr{A}^*\mathscr{A}(v)$$

习题 8.3.7. 定义 $\mathscr{A} \in \mathrm{End}(\mathbb{R}^3)$ 为 $(x, y, z)^T \longmapsto (z, 2x, 3y)^T$. 找出一个等距同构 $Q \in \mathrm{End}(\mathbb{R}^3)$ 使得 $\mathscr{A} = Q\sqrt{\mathscr{A}^*\mathscr{A}}$.

$\mathcal{M}_\varepsilon(\mathscr{A}) = \begin{pmatrix} 0 & 0 & 1 \\ 2 & 0 & 0 \\ 0 & 3 & 0 \end{pmatrix}$, $\mathcal{M}_\varepsilon(\mathscr{A}^*) = \begin{pmatrix} 0 & 2 & 0 \\ 0 & 0 & 3 \\ 1 & 0 & 0 \end{pmatrix}$. $\mathcal{M}_\varepsilon(\mathscr{A}^*\mathscr{A}) = \begin{pmatrix} 4 & & \\ & 9 & \\ & & 1 \end{pmatrix}$

令 $Q\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \frac{1}{2}\mathscr{A}\left(\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}\right) = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$, $Q\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \frac{1}{3}\mathscr{A}\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$, $Q\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \mathscr{A}\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$.

$\Rightarrow \mathcal{M}_\varepsilon(\mathscr{A}) = \begin{pmatrix} & & 1 \\ 1 & & \\ & 1 & \end{pmatrix}\begin{pmatrix} 2 & & \\ & 3 & \\ & & 1 \end{pmatrix}$ $\mathcal{M}_\varepsilon(Q) = \begin{pmatrix} & & 1 \\ 1 & & \\ & 1 & \end{pmatrix}$

习题 8.3.8. 设 $\mathscr{A} \in \mathrm{End}(V)$.

1. 证明: 若 $\mathscr{A}$ 是可逆变换, 则在极分解式 $\mathscr{A} = QS = S_1Q_1$ 中, 等距同构 Q 和 Q_1 也是由 $\mathscr{A}$ 唯一确定的.
2. 举例说明: 若 $\mathscr{A}$ 不可逆, 则极分解式 $\mathscr{A} = QS$ 中的等距同构 Q 可能不是唯一的 (S 为半正定算子).

证: 1. $\mathrm{rank}\,\mathscr{A}^*\mathscr{A} = \mathrm{rank}\,\mathscr{A} \leq \mathrm{rank}\,\sqrt{A^*A} \Rightarrow S_1, S$ 也可逆.

故 $Q = \mathscr{A}S^{-1}$, $Q_1 = S_1^{-1}\mathscr{A}$.

2. $\begin{pmatrix} 1 & & \\ & 0 & \\ & & 0 \end{pmatrix} = \begin{pmatrix} 1 & & \\ & \cos\theta & \sin\theta \\ & \sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} 1 & & \\ & 0 & \\ & & 0 \end{pmatrix}$, $\theta \in \mathbb{R}$

习题 8.3.9. 设 V 是有限维酉空间, $\mathscr{A}, \mathscr{B} \in \mathrm{End}(V)$ 为正定算子, $\mathscr{U} \in \mathrm{End}(V)$ 是酉变换. 证明: 如果 $\mathscr{A} = \mathscr{B}\mathscr{U}$, 则必然 $\mathscr{A} = \mathscr{B}$, $\mathscr{U} = I$.

It follows immediately from the uniqueness of polar factorization.

习题 8.3.10. 设 $A \in \mathbf{M}_n(\mathbb{C})$ 是正定的 Hermite 矩阵, $B \in \mathbf{M}_n(\mathbb{C})$ 是半正定的 Hermite 矩阵.

证明: AB 相似于一个对角阵 $\begin{pmatrix} d_1 & & \\ & \ddots & \\ & & d_n \end{pmatrix}$, 其中 d_i 都是非负实数.

$\exists$ positive-def A_1 and positive semi-def B_1 s.t. $A_1^2 = A$, $B_1^2 = B$.

Let $C = A_1B_1$. $CC^* = A_1B_1B_1^*A_1 = A_1BA_1 = A_1^{-1}ABA_1$

Since CC^* is Hemitian, CC^* is similar to a diagonal matrix and since B is positive semi-def, all diagonal elements can be nonnegative.

Thus, so is $AB = A_1CC^*A_1^{-1}$.